ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

Results on Nontrivial Solutions of the Schrödinger-Bopp-Podolsky System

Shuwen He[1,2,*]

[1]*School of Mathematics, Physics and Statistics, Sichuan Minzu College, Kangding, 626001, China*
[2]*Visual Computing and Virtual Reality Key Laboratory of Sichuan Province, Sichuan Normal University, Chengdu, 610068, China*
**Corresponding author: 2315244944@qq.com*

Abstract: *This article is to study a Schrödinger-Bopp-Podolsky system has steep potential well and concave-convex nonlinearities. This system is a coupled system that describes physical phenomena such as charge motion in physics, and its mathematical research involves the existence and asymptotic behavior of solutions to nonlinear partial differential equations. By combining the variational methods and the truncation technique to prove the existence and concentration behavior results of nontrivial solutions for this system. Specifically, first the defined truncation function is merged into the convolution term of the corresponding energy functional of the system and it is analyzed that the Gerami sequence of the energy functional is norm bounded. Next, this sequence is proved to be a norm bounded by the energy functional of the original system. Finally, it is proved that there are strong convergent subsequences in the obtained bounded sequence, and the main results are displayed using standard analysis methods. These results of the article have been extensively improved and expanded to previous works.*

Keywords: *Schrödinger-Bopp-Podolsky System, Variational Methods, Truncation Technique*

1. Introduction

In the current paper, study a Schrödinger-Bopp-Podolsky system defined in unbounded domain $\mathbf{R}^3$ as follows:

$$\begin{cases} -\Delta u + \lambda V(x)u + q^2 \varphi u = f(x)|u|^{p-2}u + g(x)|u|^{q-2}u, \\ -\Delta\varphi + a^2\Delta^2\varphi = 4\pi u^2, \end{cases} \tag{1}$$

where $u, \varphi : \mathbf{R}^3 \to \mathbf{R}$, $a > 0, 1 < q < 2 < p < 4,$ $V(x), f(x)$ and $g(x)$ satisfying the following assumptions:

(V1) $0 \le V(x) \in C(\mathbf{R}^3, \mathbf{R})$;

(V2) $\{V < V_0\} := \{x \in \mathbf{R}^3 : V(x) < V_0\}$ is nonempty and finite lebesgue measure for some $V_0 \ge 0$;

(V3) $\Omega = \text{int}V^{-1}(0)$ is nonempty open set and the boundary is smooth with $\overline{\Omega} = V^{-1}(0)$;

(F1) $f \in L^\infty(\mathbf{R}^3, \mathbf{R})$ and $|f|_\infty \ne 0$;

(G1) $g \in L^{\frac{p}{p-q}}(\mathbf{R}^3, \mathbf{R}^+)$ and $\max\{g(x), 0\} \not\equiv 0$.

This system typically appears in the schrödinger field $\xi = \xi(t, x)$ and its electromagnetic field. Bopp [1] and podolsky [2] independently studied the bopp podolsky electromagnetic theory, which can be seen as the second-order norm theory. From the perspective of electromagnetic fields, the Bopp Podolsky theory and Maxwell theory are indistinguishable experimentally and can be explained as effective theories for both long and short distances . The study of the existence and concentration of solutions in this system has strong physical and mathematical significance. One knows that the mathematical model of system (1) is expressed through a nonlinear partial differential system consisting of two equations, which is commonly used to describe certain physical phenomena in quantum mechanics, especially those with long-range interactions and quantum fluctuations. This system can be used to study certain specific

ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

processes in quantum field theory, such as the interaction between electrons and electromagnetic fields. Its mathematical model equations contain basic concepts in quantum mechanics, such as wave functions and potential energy. By solving this system, the ground state solution, energy solution, etc. of the quantum system can be analyzed, and the physical mechanism of quantum phenomena can be understood. These studies not only enrich our understanding of quantum phenomena, but also provide theoretical basis and reference value for further research in related fields such as mathematics and physics.

In the past few decades, the schrödinger bopp podolsky system has received widespread attention. When $\lambda V(x)$ is a normal number and the nonlinear term is $|u|^{p-2}u$ ($2<p<6$), d'avenia and siciliano they initially researched some results with and without solutions to system (1). At the same time, they also analyzed the asymptotic behavior results of nontrivial solutions. Subsequently, if nonlinear term is $\mu|u|^{p-1}u+|u|^4u$, where $\mu>0$, $2<p<5$, li et al. Considered the critical situation of the system (1), they got the solutions of system (1) by combining the pohozaev nehari manifold method with monotonicity analysis techniques. Later, chen et al. Investigated the existence results of ground state solutions for system (1) under some relaxed conditions for λv and $f+g$. In addition, they also minutely provided the maximum minimization feature of the ground state energy, jia and li also discussed the non autonomous system, they cleverly used the nehari manifold and split lemma to prove that system (1) has a solution. For more results on this problem, see [3-7] and so on.

Motivated by the works mentioned above [5,7], the main objective of this article is to study the system (1). More precisely, follow the variational methods and use an interesting truncation the technique to prove the existence and concentration behavior of the nontrivial solutions of system (1) when q is small and λ is large enough, this is significantly different from the previous works.

Now let's state the main results.

Theorem 1.1 Under the conditions that $(\mathrm{V1})-(\mathrm{V3})$, (F1) and (G1) are established. Then there exist $\hat{q},\hat{\lambda},\Pi>0$ such that $q\in(0,\hat{q})$, $\lambda>\hat{\lambda}$ and $0<|g|_{\frac{p}{p-q}}<\Pi$, system (1) possess at least one nontrivial solution u. Moreover, there exist $\beta,\eta>0$ such that

$$0<\|u\|_{\lambda}\le\beta \quad \text{and} \quad \eta\le J_{\lambda,q}(u).$$

Theorem 1.2 Under the conditions that $(\mathrm{V1})-(\mathrm{V3})$, (F1) and (G1) are established. Let u_λ be a solution for system (1) given by Theorem 1.1. Going if necessary a subsequence, then for every $q\in(0,\hat{q})$ and $0<|g|_{\frac{p}{p-q}}<\Pi$, $u_\lambda\to u_0$ in $H^1(\mathbf{R}^3)$ as $\lambda\to\infty$, where u_0 is a nontrivial solution of

$$\begin{cases}-\Delta u+q^2\varphi_u u=f(x)|u|^{p-2}u+g(x)|u|^{q-2}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}$$

Remark 1.1 1) The nonlinear terms f and g no longer use the Ambrosetti Rabinowitz condition (A-R condition), but instead use a weakened condition to get some results about solutions for System (1);

2) when $V(x)$ is a steep potential and $f+g$ is a nonlinear concave-convex with $f(x)\,|u|^{q-2}u+G(x)\,|u|^{p-2}u$, where $1<q<2<p<4$, this paper discuss the existence and concentration of solutions in system (1), which has almost never been considered before and can supplement and generalize existing research results;

3) $V(x)$ is a steep potential and $f+g$ is a more general continuous nonlinear term without differentiability, considering the existence and concentration of solutions in system (1) from this more general situation provides certain reference value for subsequent research on such problems.

2. Preliminaries

Before starting the proof of the Theorems, there mainly state that the main preparatory knowledge and some useful results for our problem. $|\cdot|_p$ is the ordinary norm of $L^p(\mathbf{R}^3)$ for all $1\le p\le\infty$, E^* denotes the dual space of E, $o(1)$ is any quantity which tends to zero when $n\to\infty$. Moreover, if the subsequence of $\{u_n\}$ is taken, it will be denoted again as $\{u_n\}$. $H^1(\mathbf{R}^3)$ denotes a Sobolev space and $\|u\|_{H^1}$ is a norm. Let

ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

$$E=\{u\in H^1(\mathbf{R}^3):\int_{\mathbf{R}^3}V(x)u^2\mathrm{d}x<\infty\}$$

be endowed with

$$\sqrt{\langle u,v\rangle}=\left(\int_{\mathbf{R}^3}(\nabla u\nabla v+V(x)uv)\mathrm{d}x\right)^{\frac{1}{2}}=\|u\|.$$

For $\lambda>0$, here also need the following two definitions

$$\sqrt{\langle u,v\rangle_\lambda}=\left(\int_{\mathbf{R}^3}(\nabla u\nabla v+\lambda V(x)uv)\mathrm{d}x\right)^{\frac{1}{2}}=\|u\|_\lambda.$$

If $\lambda\geq 1$, then $\|u\|\leq\|u\|_\lambda$. Set $E_\lambda=\left(E,\|\cdot\|_\lambda\right)$. Then it's easy to get the following lemma.

Lemma 2.1 Under the conditions that $(\mathrm{V1})-(\mathrm{V3})$ are established. Then the embedding E_λ is continuously embedded into $L^s(\mathbf{R}^3)$ $(\forall s\in[2,6])$ and $\lambda\geq 1$. Therefore, there is $ds>0$ (not related to $\lambda\geq 1$) derives

$$|u|_s\leq d_s\|u\|\leq d_s\|u\|_\lambda,\quad \text{for all } u\in E. \tag{2}$$

After analysis, one can easily determine that system (1) the same as

$$-\Delta u+\lambda V(x)u+q^2\left(\frac{1-e^{-\frac{|x|}{a}}}{|x|}*u^2\right)u=f(x)|u|^{p-2}u+g(x)|u|^{q-2}u,\quad x\in\mathbf{R}^3,$$

It's solutions of this equation are the critical points of the functional $J_{\lambda,q}\in C^1(E_\lambda,\mathbf{R})$ given by

$$J_{\lambda,q}(u)=\frac{1}{2}\|u\|_\lambda^2+\frac{q^2}{4}\int_{\mathbf{R}^3}\varphi_u u^2\mathrm{d}x-\frac{1}{p}\int_{\mathbf{R}^3}f(x)|u|^p\,\mathrm{d}x-\frac{1}{q}\int_{\mathbf{R}^3}g(x)|u|^q\,\mathrm{d}x.$$

In order to prove the main results, now move to consider the truncated functional

$$J_{\lambda,q}^\beta(u)=\frac{1}{2}\|u\|_\lambda^2+\frac{q^2}{4}\varphi\left(\frac{\|u\|_\lambda^2}{\beta^2}\right)\int_{\mathbf{R}^3}\varphi_u u^2\mathrm{d}x-\frac{1}{p}\int_{\mathbf{R}^3}f(x)|u|^p\,\mathrm{d}x-\frac{1}{q}\int_{\mathbf{R}^3}g(x)|u|^q\,\mathrm{d}x,$$

where φ is defined as

$$\varphi\left(\frac{\|u\|_\lambda^2}{\beta^2}\right)=\begin{cases}1, & \text{if } \|u\|_\lambda^2\in(0,\ \beta^2],\\ 0, & \text{if } \|u\|_\lambda^2\in[2\beta^2,+\infty).\end{cases}$$

One can easily verify that $J_{\lambda,q}^\beta\in(E_\lambda,\mathbf{R})$.

Remark 2.1 Proposed technological route:

1) Define the usual Sobolev space inner product and norm, and use the Ekeland variational Principle to construct the energy functional corresponding to the nonlinear Schrödinger-Bopp-Podolsky system;

2) Using critical point theory, verify that the solution of the system corresponds one-to-one With the critical points of the functional;

3) Verify the defined functional satisfies the geometric structure of Mountain Pass;

4) By constructing some useful inequalities to verify the existence of minimization sequences, the functional energy value can reach the defined minimum energy;

5) Calculate the range of values for the lowest energy value and show the boundedness of the minimization sequence;

6) Standard discussion methods and analytical principles were used to study the existence and

ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

interesting states of solutions on the system.

3. Proof of Theorems 1.1 and 1.2

This section mainly prove Theorems 1.1 and 1.2. First of all, define a cut-off function $\varphi \in C^1([0,\infty), \mathbf{R})$ satisfying

$$\begin{cases} \varphi(t)=1, & \text{if } 0\le t\le 1,\\ 0\le \varphi(t)\le 1, & \text{if } 1<t<2,\\ \varphi(t)=0, & \text{if } t\ge 2,\\ \max\limits_{t>0} |\varphi'(t)|\le 2, \ \varphi'(t)\le 0, & \text{if } t>0. \end{cases}$$

Lemma 3.1 Under the conditions that $(\text{V1})-(\text{V3})$, (F1) and (G1) are established. Then for all $\lambda \ge 1$,

(A1) there exist $\Pi, \rho, \eta > 0$ such that for all $0<|g|_{\frac{p}{p-q}}<\Pi$ and $\|u\|_\lambda=\rho$, one has $J_{\lambda,q}^{\beta}(u)\ge \eta$;

(A2) there exist $q_1>0$ and $e_0\in C_0^\infty(\Omega)$ with $|\nabla e_0|_2>\rho$ such that $J_{\lambda,q}^{\beta}(e_0)<0$ for all $q\in(0,q_1)$.

Proof Using (F1), (G1), (2) and Hölder inequality, one has

$$\begin{aligned} J_{\lambda,q}^{\beta}(u) &\ge \frac{1}{2}\|u\|_\lambda^2-\frac{1}{p}\int_{\mathbf{R}^3} f(x)|u|^p\,\mathrm{d}x-\frac{1}{q}\int_{\mathbf{R}^3} g(x)|u|^q\,\mathrm{d}x\\ &\ge \frac{1}{2}\|u\|_\lambda^2-\frac{d_p^p}{p}|f|_\infty\|u\|_\lambda^p-\frac{d_p^q}{q}|g|_{\frac{p}{p-q}}\|u\|_\lambda^q\\ &=\|u\|_\lambda^q\left(\frac{1}{2}\|u\|_\lambda^{2-q}-\frac{d_p^p}{p}|f|_\infty\|u\|_\lambda^{p-q}-\frac{d_p^q}{q}|g|_{\frac{p}{p-q}}\right). \end{aligned}$$

One can easily deduce that $H(t):=\frac{1}{2}t^{2-q}-\frac{d_p^p}{p}|f|_\infty t^{p-q}$ has a unique maximum point at

$$t_0=\left(\frac{p(2-q)}{2(p-q)d_p^p|f|_\infty}\right)^{\frac{1}{p-2}},$$

and its maximum is

$$H(t_0):=\frac{p-2}{2(p-q)}\left(\frac{p(2-q)}{2(p-q)d_p^p|f|_\infty}\right)^{\frac{2-q}{p-2}}>0.$$

Thus, choose $t_0=\rho=\|u\|_\lambda>0$ such that $H(\rho)>0$. By simple calculation, there exists $\Pi>0$ such that $0<|g|_{\frac{p}{p-q}}<\Pi$,

$$J_{\lambda,q}^{\beta}(u)\ge \rho^q\left(H(\rho)-\frac{d_p^q}{q}|g|_{\frac{p}{p-q}}\right)\ge \frac{\rho^q}{4}H(\rho):=\eta>0.$$

In order to prove (A2), define the function $J_\lambda\in(E_\lambda,\mathbf{R})$ by

ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

$$J_\lambda(u)=\frac{1}{2}\|u\|_\lambda^2-\frac{1}{p}\int_{\mathbf{R}^3}f(x)|u|^p\,\mathrm{d}x-\frac{1}{q}\int_{\mathbf{R}^3}g(x)|u|^q\,\mathrm{d}x.$$

Thus, take a positive function $e\in C_0^\infty(\Omega)$ reach to

$$J_\lambda(le)=\frac{l^2}{2}\int_\Omega|\nabla e|^2\,\mathrm{d}x-\frac{l^p}{p}\int_\Omega f(x)|e|^p\,\mathrm{d}x-\frac{l^q}{q}\int_\Omega g(x)|e|^q\,\mathrm{d}x\to-\infty \text{ as } l\to\infty.$$

Therefore, there exist $e_0\in C_0^\infty(\Omega)$ with $|e_0|_2>\rho$, one has $J_\lambda(e_0)\le -1$. Then, for all $q\in(0,q_1)$, there is $q_1>0$ such that

$$\begin{aligned}J_{\lambda,q}^\beta(e_0)&=J_\lambda(e_0)+\frac{q^2}{4}\varphi\left(\frac{\|e_0\|_\lambda^2}{\beta^2}\right)\int_{\mathbf{R}^3}\varphi_{e_0}e_0^2\mathrm{d}x\\&\le -1+\frac{q^2}{4}\int_{\mathbf{R}^3}\varphi_{e_0}e_0^2\mathrm{d}x\\&\le -1+\frac{q^2}{4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{e_0^2(x)e_0^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y\\&\le -1+\frac{q^2}{4}C|e_0|_{\frac{12}{5}}^4\\&<0,\end{aligned}$$

Now defining energy value

$$c_{\lambda,q}^\beta=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J_{\lambda,q}^\beta(\gamma(t))$$

with $\Gamma=\{\gamma\in C([0,1],E_\lambda):\gamma(0)=(0),\gamma(1)=e_0\}$. Then by lemma 3.1 and the variant of the mountain Pass Theorem [8], one deduces that for all $\lambda\ge 1$ and $q\in(0,q_1)$, there is a Gerami sequence $\{u_n\}\subset E_\lambda$ at the energy value $c_{\lambda,q}^\beta$>0 and the following relationships are found

$$J_{\lambda,q}^\beta(u_n)\to c_{\lambda,q}^\beta\ge\eta>0 \text{ and } (1+\|u_n\|_\lambda)\|(J_{\lambda,q}^\beta)'(u_n)\|_{E_\lambda^*}\to 0. \tag{3}$$

Lemma 3.2 Under the conditions that $(\mathrm{V1})-(\mathrm{V3})$, (F1) and (G1) are established. Then for all $\lambda\ge 1$ and $q\in(0,q_1)$, there is $M>0$ such that $c_{\lambda,q}^\beta\le M$.

Proof Let $e_0\in C_0^\infty(\Omega)$. From Lemma 2.1 and Hölder inequality, one gets

$$\begin{aligned}J_{\lambda,q}^\beta(le_0)&=\frac{l^2}{2}\int_\Omega|\nabla e_0|^2\,\mathrm{d}x+\frac{q^2l^4}{4}\varphi\left(\frac{l^2\|e_0\|_\lambda^2}{\beta^2}\right)\int_\Omega\varphi_{e_0}e_0^2\mathrm{d}x\\&\quad-\frac{l^p}{p}\int_\Omega f(x)|e_0|^p\,\mathrm{d}x-\frac{l^q}{q}\int_\Omega g(x)|e_0|^q\,\mathrm{d}x\\&\le\frac{l^2}{2}\int_\Omega|\nabla e_0|^2\,\mathrm{d}x+\frac{q^2l^4}{4}\int_\Omega\varphi_{e_0}e_0^2\mathrm{d}x-\frac{l^p}{p}\int_\Omega f(x)|e_0|^p\,\mathrm{d}x+\frac{l^q}{q}|g|_{L^{\frac{p}{p-q}}(\Omega)}\left(\int_\Omega|e_0|^p\,\mathrm{d}x\right)^{\frac{q}{p}}\\&\le\frac{l^2}{2}\int_\Omega|\nabla e_0|^2\,\mathrm{d}x+\frac{\hat{q}^2l^4}{4}C|e_0|_{\frac{12}{5}}^4-\frac{l^p}{p}\int_\Omega f(x)|e_0|^p\,\mathrm{d}x+\frac{l^q}{q}|g|_{L^{\frac{p}{p-q}}(\Omega)}\left(\int_\Omega|e_0|^p\,\mathrm{d}x\right)^{\frac{q}{p}}.\end{aligned}$$

ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

Thus, there exists $M>0$ (not related to β,λ,b) such that

$$c_{\lambda,q}^{\beta}=\max_{t\in[0,1]}J_{\lambda,q}^{\beta}(te_0)\le M .$$

Lemma 3.3 Under the conditions that $(\mathrm{V1})-(\mathrm{V3})$, (F1) and (G1) are established. If $\{u_n\}\subset E_\lambda$ is a Cerami sequence satisfying (3). Passing to a subsequence, there exist $\beta>0$ and $q_2>0$ such that for any $\lambda\ge 1, q\in(0,\hat{q})$ with $\hat{q}:=\min\{q_1,q_2\}$, there holds $\|u_n\|_\lambda\le\beta.$

Proof We need to split the following cases.

Case 1. If $\|u_n\|_\lambda>\sqrt{2}\beta$, by Lemma 3.2, one has

$$M+1\ge c_{\lambda,q}^{\beta}+1\ge J_{\lambda,q}^{\beta}(u_n)-\frac{1}{p}\left\langle (J_{\lambda,q}^{\beta})'(u_n),u_n\right\rangle$$

$$=\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\lambda^2-\left(\frac{q^2}{p}-\frac{q^2}{4}\right)\varphi\left(\frac{\|u_n\|_\lambda^2}{\beta^2}\right)\int_{\mathbf{R}^3}\varphi_{u_n}u_n^2\mathrm{d}x$$

$$-\frac{q^2}{2p\beta^2}\varphi'\left(\frac{\|u_n\|_\lambda^2}{\beta^2}\right)\|u_n\|_\lambda^2\int_{\mathbf{R}^3}\varphi_{u_n}u_n^2\mathrm{d}x-\left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbf{R}^3}g(x)|u_n|^q\,\mathrm{d}x$$

$$\ge\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\lambda^2-\left(\frac{1}{q}-\frac{1}{p}\right)d_p^q|g|_{\frac{p}{p-q}}\|u_n\|_\lambda^q$$

For n large enough, which is contradiction when $\beta>0$ is sufficiently large.

Case 2. $\beta<\|u_n\|_\lambda\le\sqrt{2}\beta.$

$$\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_\lambda^2-\frac{1}{p}\|(J_{\lambda,q}^{\beta})'(u_n)\|_{E_\lambda^*}\|u_n\|_\lambda$$

$$\le J_{\lambda,q}^{\beta}(u_n)+\left(\frac{q^2}{p}-\frac{q^2}{4}\right)\varphi\left(\frac{\|u_n\|_\lambda^2}{\beta^2}\right)\int_{\mathbf{R}^3}\varphi_{u_n}u_n^2\mathrm{d}x$$

$$+\frac{q^2}{2p\beta^2}\varphi'\left(\frac{\|u_n\|_\lambda^2}{\beta^2}\right)\|u_n\|_\lambda^2\int_{\mathbf{R}^3}\varphi_{u_n}u_n^2\mathrm{d}x+\left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbf{R}^3}g(x)|u_n|^q\,\mathrm{d}x$$

$$\le J_{\lambda,q}^{\beta}(u_n)+\left(\frac{q^2}{p}-\frac{q^2}{4}\right)C|u_n|_{\frac{12}{5}}^4+\left(\frac{1}{q}-\frac{1}{p}\right)d_p^q|g|_{\frac{p}{p-q}}\|u_n\|_\lambda^q$$

$$\le J_{\lambda,q}^{\beta}(u_n)+4Cq^2\beta^4+C\beta^q .$$

It follows from $J_{\lambda,q}^{\beta}(u_n)\to c_{\lambda,q}^{\beta}$ and Lemma 3.2 that

$$C\beta^2-\beta\le 2M+4Cq^2\beta^4+C\beta^q ,$$

which is a contradiction if $q_2:=\frac{1}{\beta^2}>0$, $q\in(0,\hat{q})$ for β large enough and $\hat{q}:=\min\{q_1,q_2\}$.

Remark 3.1 If the sequence $\{u_n\}\subset E_\lambda$ is given in Lemma 3.3, then $\{u_n\}$ is also a bounded Gerami sequence of $J_{\lambda,q}$ satisfying $\|u_n\|_\lambda\le\beta.$ By the definition of the truncation function φ, one can see that

ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

$$J_{\lambda,q}(u_n)\to c_{\lambda,q}^{\beta} \quad \text{and} \quad (1+\|u_n\|_\lambda)\|(J_{\lambda,q})'(u_n)\|_{E_\lambda^*}\to 0. \tag{4}$$

Lemma 3.4 Under the conditions that $(V1)-(V3)$, (F1) and (G1) are established. If $\{u_n\}\subset E_\lambda$ is a Cerami sequence satisfying (4). Going if necessary a subsequence, there is $\lambda_1>0$, such that for each $\lambda\ge\hat{\lambda}$ with $\hat{\lambda}=\max\{\lambda_1,1\}$, then $u_{n_k}\to u\,(\{u_{n_k}\}\subseteq\{u_n\})$ in E_λ for all $q\in(0,\hat{q})$.

Proof From Lemma 3.3, take another subsequence if necessary, one has $\|u_n\|_\lambda\le\beta$. Then assume that there exists $u\in E_\lambda$ such that $u_n\to u$ weakly in E_λ, $u_n\to u$ in $L^s_{loc}(\mathbf{R}^3)$ $(\forall s\in[2,6))$ and $u_n\to u$ a.e. on $\mathbf{R}^3$. If u is a critical point of $J_{\lambda,q}$, then $\langle J'_{\lambda,q}(u),v\rangle=0$, thus

$$\langle J'_{\lambda,q}(u),u\rangle=\|u\|_\lambda^2+q^2\int_{\mathbf{R}^3}\varphi_u u^2\mathrm{d}x-\int_{\mathbf{R}^3}f(x)|u|^p\,\mathrm{d}x-\int_{\mathbf{R}^3}g(x)|u|^q\,\mathrm{d}x=0.$$

Define $v_n=u_n-u$. Hence one can get that $v_n\to 0$ weakly in E_λ and $\|v_n\|_\lambda\le 2\beta$. From (2), (V2) and Hölder inequality, there is σ ∈ (0, 1) such that $\frac{1}{p}=\frac{\sigma}{2}+\frac{1-\sigma}{6}$ and

$$\int_{\mathbf{R}^3}f(x)|v_n|^p\,\mathrm{d}x\le|f|_\infty\left(\int_{\mathbf{R}^3}|v_n|^2\,\mathrm{d}x\right)^{\frac{p\sigma}{2}}\left(d_6^6\|v_n\|_\lambda^6\right)^{\frac{p(1-\sigma)}{6}}$$

$$\le\left(\frac{1}{\lambda V_0}\right)^{\frac{p\sigma}{2}}d_6^{p(1-\sigma)}|f|_\infty\|v_n\|_\lambda^p+o(1). \tag{5}$$

By (G1), $|v_n|^q\to 0$ weakly in $L^{\frac{p}{q}}(\mathbf{R}^3)$ and the Brézis-Lieb Lemma [9], one has

$$\begin{cases}\int_{\mathbf{R}^3}f(x)|v_n|^p\,\mathrm{d}x=\int_{\mathbf{R}^3}f(x)|u_n|^p\,\mathrm{d}x-\int_{\mathbf{R}^3}f(x)|u|^p\,\mathrm{d}x+o(1),\\ o(1)=\int_{\mathbf{R}^3}g(x)|v_n|^q\,\mathrm{d}x=\int_{\mathbf{R}^3}g(x)|u_n|^q\,\mathrm{d}x-\int_{\mathbf{R}^3}g(x)|u|^q\,\mathrm{d}x+o(1).\end{cases} \tag{6}$$

Thus, using (5) and (6) leads to

$$o(1)=\langle J'_{\lambda,q}(u_n),u_n\rangle-\langle J'_{\lambda,q}(u),u\rangle$$

$$=\|v_n\|_\lambda^2+q^2\int_{\mathbf{R}^3}\varphi_{v_n}v_n^2\mathrm{d}x-\int_{\mathbf{R}^3}f(x)|v_n|^p\,\mathrm{d}x-\int_{\mathbf{R}^3}g(x)|v_n|^q\,\mathrm{d}x+o(1)$$

$$\ge\|v_n\|_\lambda^2-\int_{\mathbf{R}^3}f(x)|v_n|^p\,\mathrm{d}x-\int_{\mathbf{R}^3}g(x)|v_n|^q\,\mathrm{d}x+o(1)$$

$$\ge\|v_n\|_\lambda^2\left[1-\left(\frac{1}{\lambda V_0}\right)^{\frac{p\sigma}{2}}d_6^{p(1-\sigma)}|f|_\infty(2\beta)^{p-2}\right]+o(1).$$

Obviously, there exists $\lambda_1>0$, such that for each $\lambda\ge\hat{\lambda}$ with $\hat{\lambda}=\max\{\lambda_1,1\}$, then $\{v_n\}\to 0$ in E_λ.

Prove the Theorem 1.1 Set β be given by above. By Lemmas 3.1-3.3 and Remark 3.1, there exist constants $\hat{q},\Pi>0$ such that for all $q\in(0,\hat{q})$ and $0<|g|_{\frac{p}{p-q}}<\Pi$, $J_{\lambda,q}$ has a Cerami sequence $\{u_n\}\subset E_\lambda$ at the mountain pass level $c_{\lambda,q}^{\beta}$ for all $\lambda\ge 1$, up to a subsequence if necessary, $\{u_n\}$ satisfies

ISSN 2616-5805 Vol. 6, Issue 2: 113-120, DOI: 10.25236/AJMS.2025.060215

$$\sup_{n\in N^+} \| u_n \|_\lambda \le \beta, \ J_{\lambda,q}(u_n) \to c_{\lambda,q}^\beta \ \text{ and } \ (1+\| u_n \|_\lambda) \| (J_{\lambda,q})'(u_n) \|_{E_\lambda^*} \to 0.$$

It follows from Lemma 3.4 that there exists $\hat{\lambda} > 0$ such that for each $\lambda \ge \hat{\lambda}$, $\{u_n\}$ has a convergent subsequence in E_λ for all $q \in (0, \hat{q})$. Then assume that $u_n \to u$ as $n \to \infty$, and thus

$$0 < \| u \|_\lambda \le \beta, \ J_{\lambda,q}(u) = c_{\lambda,q}^\beta \ \text{ and } \ (J_{\lambda,q})'(u) = 0.$$

Consequently, *u* is a nontrivial solution for system (1).

Prove the Theorem 1.2 The proof draw support from Theorem 1.3 in [10], one can easily finish this theorem.

4. Conclusions

This article is based on the strong background significance of Schrödinger-Bopp-Podolsky systems in physics, using Ekeland's variational principle and critical point theory, as well as studying the existence and concentration phenomena of solutions for such systems under more optimized conditions. It has strong application and academic value. More precisely, when $q \to 0$ and $\lambda \to \infty$, by using the variational methods and truncation technique, this paper accurately harvest the existence and concentration of a nontrivial solutions to system (1) involving steep potential well and concave-convex nonlinear terms. The provided results not only improve and expand the research published in [5], but also provide a good idea for studying the existence and behavior of solutions to this type of problems.

Acknowledgements

This project is supported by funding of Visual Computing and Virtual Reality Key Laboratory of Sichuan Province "Research on the Existence and Concentration of Solutions in Nonlinear Schrödinger -Bopp-Podolsky Systems" (No. SCVCVR2024.01VS) and the Funding Project of Mathematics and Finance Research Center, Key Research Base of Dazhou Federation of Social Sciences "Research on Fractional Order Duopoly Game Model with Corporate Social Responsibility" (No. SCMF202407).

References

[1] Bopp F. Eine lineare theorie des elektrons[J]. Annalen der Physik, 1940, 430(5): 345-384.
[2] Podolsky B. A generalized electrodynamics part I-non-quantum[J]. Physical Review, 1942, 62 (1-2): 68-71.
[3] Yang J, Chen H, Liu S. The existence of nontrivial solution of a class of Schrödinger-Bopp-Podolsky system with critical growth[J]. Boundary Value Problems, 2020, 2020(1): 144.
[4] Wang L, Chen H, Liu S. Existence and multiplicity of sign-changing solutions for a Schrödinger-Bopp-Podolsky system[J], Topological Methods in Nonlinear Analysis, 2022, (59): 913-940.
[5] Zhang Q. Sign-changing solutions for Schrödinger-Bopp-Podolsky system with general nonlinearity[J]. Zeitschrift für angewandte Mathematik und Physik, 2022, 73(6): 235.
[6] Peng X. Existence and multiplicity of solutions for the Schrödinger-Bopp-Podolsky system[J]. Bulletin of the Malaysian Mathematical Sciences Society, 2022, 45(6): 3423-3468.
[7] Zhu Q, Chen C, Yuan C. Schrödinger-Bopp-Podolsky System with Steep Potential Well[J]. Qualitative Theory of Dynamical Systems, 2023, 22(4): 140.
[8] Ekeland I. Convexity methods in Hamiltonian mechanics[M]. Springer Science & Business Media, 2012.
[9] Willem M. Minimax Theorems[M]. Birkhäuser, Boston, 1996.
[10] He S, Wen X. Existence and concentration of solutions for a Kirchhoff-type problem with sublinear perturbation and steep potential well[J]. AIMS Mathematics, 2023, 8(3): 6432-6446.