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The Application and Stability Analysis of Stochastic Differential Equations in Financial Mathematics

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Abstract: *As a core tool in the field of financial mathematics, stochastic differential equations (SDEs) not only profoundly depict the complexity and variability of financial markets, but also play an irreplaceable role in the high-end financial engineering of option pricing. SDEs effectively simulate random fluctuations, jump risks, and unpredictable factors in the market by introducing random terms such as Brownian motion or more general Lévy processes, providing a powerful framework for accurate modeling of dynamic processes in financial markets. In the field of option pricing, the application of SDEs is particularly crucial. Taking the Black-Scholes pricing model as an example, it is based on the assumption of geometric Brownian motion and obtains an explicit formula for the price of European options by solving a specific SDE. In addition, the stability analysis of SDEs in option pricing is also one of the research hotspots. Stability is not only related to the reliability of model prediction results, but also directly affects the effectiveness of trading strategies. The in-depth application and stability research of SDEs in financial mathematics, especially in option pricing, is of great significance for promoting the development and innovation of financial markets.*

Keywords: *Stochastic differential equation; Financial Mathematics; Application; stability analysis*

1. Introduction

With the acceleration of global economic integration and the continuous deepening of financial market innovation, traditional basic options are no longer able to meet the increasingly diverse needs of investors [1]. In order to attract more capital inflows, financial institutions continue to innovate and design a series of new option products that meet the special needs of investors, such as Asian options, barrier options, look back options, etc [2]. These new options exhibit more complex characteristics in structure, pricing, and risk management, posing higher challenges to the fields of financial mathematics and engineering [3]. As the core issue of option trading, the accuracy of option pricing directly affects the distribution of interests among market participants and the overall stability of the market [4]. Since Black and Scholes proposed the famous B-S model in 1973, probability based stochastic finance methods have become the mainstream of option pricing research.

The B-S model successfully derived the pricing formula for European options by assuming that the underlying asset price follows geometric Brownian motion, which is only affected by random fluctuations, laying a solid foundation for the development of financial derivative pricing theory [5]. However, with the continuous development and complexity of financial markets, the limitations of the B-S model have gradually become apparent. In the actual market, the price of the underlying asset is often influenced by multiple factors, including macroeconomic environment, market sentiment, policy changes, etc. These factors are often difficult to accurately describe using a single stochastic process [6]. Therefore, in order to more closely depict the dynamic changes in financial markets and improve the accuracy and robustness of option pricing, researchers have begun to focus on more complex mathematical models, such as SDEs [7]. As a powerful tool for describing the dynamic changes of stochastic processes, SDEs can simultaneously consider multiple factors such as deterministic trends, random fluctuations, and possible jump risks in asset prices, providing a more flexible and comprehensive framework for modeling financial markets [8].

In the field of option pricing, the application of SDEs is not limited to extending and modifying traditional B-S models, but also to exploring pricing mechanisms for new option products. For example, for retrospective options with path dependence characteristics, their pricing needs to consider the highest or lowest value of the underlying asset price reached during the option's validity period, which makes traditional pricing methods difficult to apply. By constructing appropriate SDEs models and

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combining them with numerical solving techniques, the reasonable price of the look back option can be accurately estimated. In addition, the stability of SDEs in option pricing is also one of the research focuses. Stability is not only related to the reliability of model prediction results, but also directly affects the effectiveness of trading strategies and the overall stability of the market. Therefore, when constructing SDEs models, it is necessary to fully consider the stability conditions of the model to ensure that it can maintain good predictive performance and robustness in the face of market volatility and uncertainty.

2. The Application of SDEs in Financial Mathematics

2.1. Black-Scholes pricing model

As one of the iconic achievements in the field of financial mathematics, the Black-Scholes pricing model's core function is to provide an accurate and widely applicable pricing framework for European options [9]. This model is rooted in the fundamental assumption that stock prices follow geometric Brownian motion, which cleverly captures the randomness and trendiness inherent in stock price changes by introducing a differential equation (SDE) containing random terms to quantify this process. Specifically, it indicates that at any given moment t, the evolution of stock prices S_t follows a specific mathematical law, which not only reflects the volatility of asset prices but also reveals the essence of market uncertainty.

$$fracdS_tS_t = \mu dt + \sigma dW_t \tag{1}$$

Among them, μ is the expected return rate of the stock, σ is the volatility, and W_t is the standard Brownian motion.

When constructing financial market models, we often make simplifying assumptions that there are only two basic securities in the market: risk-free assets (such as bonds) and risky assets (such as stocks). For risk-free assets, their price dynamics follow an intuitive equation: $dM(t) = rM(t)dt$, where r represents the risk-free interest rate in the market, which is a constant time preference rate, and $M(t)$ represents the price of t risk-free assets at any given time. This equation directly reflects the characteristic of risk-free assets appreciating at a fixed interest rate over time. On the other hand, the price fluctuations of risk assets (such as stocks) are more complex, as they are influenced by market volatility, investor sentiment, and various uncertain factors. To characterize this characteristic, we introduce a differential equation to describe its price behavior, which comprehensively considers the rate of price change over time, random volatility components, and possible other influencing factors.

$$dS(t) = S(t)\left(\mu dt + \sigma dW_t^H\right), S(0) = S_0, 0 \le t \le T \tag{2}$$

Among them, $S(t)$ represents the price of risk assets, μ represents the expected rate of return of risk asset prices, $\sigma > 0$ is the volatility of risk asset prices, and $\{W_t^H, t \ge 0\}$ is the fractional Brownian motion defined on the complete probability space (Ω, F, P).

2.2. Portfolio Optimization

Portfolio optimization is one of the core issues in the financial field, aimed at maximizing expected returns or minimizing risks through scientific and rational allocation of assets in complex and uncertain market environments [10]. In this challenge, the combination of stochastic control theory and stochastic differential equations has demonstrated strong application value. Random control theory, as an important branch of modern control theory, excels in dealing with dynamic systems containing random disturbances. By designing optimal controllers, it predicts and controls the random deviations of the system, achieving optimal system performance. And stochastic differential equations provide precise mathematical tools for describing price changes and fluctuations in financial markets, capturing the randomness, correlation, and volatility in asset prices. To apply the combination of the two to portfolio optimization problems, it is first necessary to abstract the process of constructing the investment portfolio as a stochastic control system.

In this system, the state variable can be defined as the investment ratio of various assets, while the control variable is the operation of adjusting these investment ratios. By constructing corresponding stochastic differential equation models, the dynamic changes in asset prices under different market conditions can be characterized. Next, using stochastic optimal control theory, especially methods such as linear quadratic Gaussian (LQG) control, the optimal control strategy that maximizes expected returns or minimizes risks at a given risk level can be solved. This process not only involves complex mathematical deductions, but also requires simulation experiments combined with market data to verify the effectiveness and robustness of the strategy. Ultimately, by combining stochastic control theory with stochastic differential equations, investors can construct and optimize their investment portfolios more scientifically and accurately to cope with market uncertainty and achieve asset preservation and appreciation. This method not only enhances the scientific and accurate nature of investment decisions, but also provides new ideas and tools for financial risk management.

3. Experimental Result

3.1. Stability Analysis

In the field of financial mathematics, stability analysis is the cornerstone of ensuring model reliability and prediction accuracy. For models involving stochastic differential equations, stability analysis is particularly crucial as it directly relates to the long-term behavioral characteristics of the model solutions. A stable system means that its solution can remain stable or fluctuate around a stable state for a long time, which is of great significance for financial market prediction and investment decision-making. When conducting stability analysis of stochastic differential equations, researchers mainly focus on two aspects: one is theoretical stability analysis, which determines whether the solution has some form of stability by analyzing factors such as the coefficients and boundary conditions of the equation; The second is numerical stability analysis, which is a problem that needs to be considered when using numerical methods to solve stochastic differential equations.

Numerical stability is directly related to the accuracy of simulation results. Due to the high complexity and randomness of financial markets, numerical methods must be able to accurately capture the effects of random terms while maintaining the stability of the solution during long-term simulations, avoiding numerical divergence or unnecessary oscillations. This not only requires the selection of appropriate numerical algorithms, but also fine tuning of algorithm parameters to ensure that the error between the numerical solution and the true solution is within a controllable range. Therefore, in the practice of financial mathematics, stability analysis is not only a necessary part of theoretical research, but also an important aspect that cannot be ignored in practical applications. Only through comprehensive stability analysis can the effectiveness and reliability of stochastic differential equation models be ensured in financial forecasting and risk management.

3.2. Experimental Analysis

Figure 1 visually illustrates the significant advantages of the SDEs model constructed in this article over the traditional B-S model in market price prediction. In the figure, by comparing the differences between the actual and observed market price predictions of two models at multiple time points, it can be clearly seen that the price prediction accuracy of our model is higher and closer to the real market volatility. This advantage is mainly attributed to the comprehensive consideration of multiple key factors in the construction process of the model in this article. Firstly, the model in this article not only captures the basic trend of price changes over time, but also deeply analyzes the random volatility components in the price change rate, which enables the model to more accurately reflect the randomness and uncertainty commonly present in financial markets.

Secondly, the model in this article innovatively incorporates possible other influencing factors, such as market sentiment, policy changes, macroeconomic indicators, etc. The introduction of these factors further enhances the explanatory power and predictive ability of the model. By comparing Figure 1, we can clearly see that the SDEs model constructed in this paper demonstrates higher prediction accuracy and stronger adaptability in the face of complex and ever-changing financial market environments. This discovery not only provides new ideas and methods for predicting financial markets, but also offers more reliable and effective tools for investors and financial institutions in risk management, asset allocation, and other areas. Therefore, the model presented in this article has significant theoretical

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value and practical significance in the field of financial market forecasting.

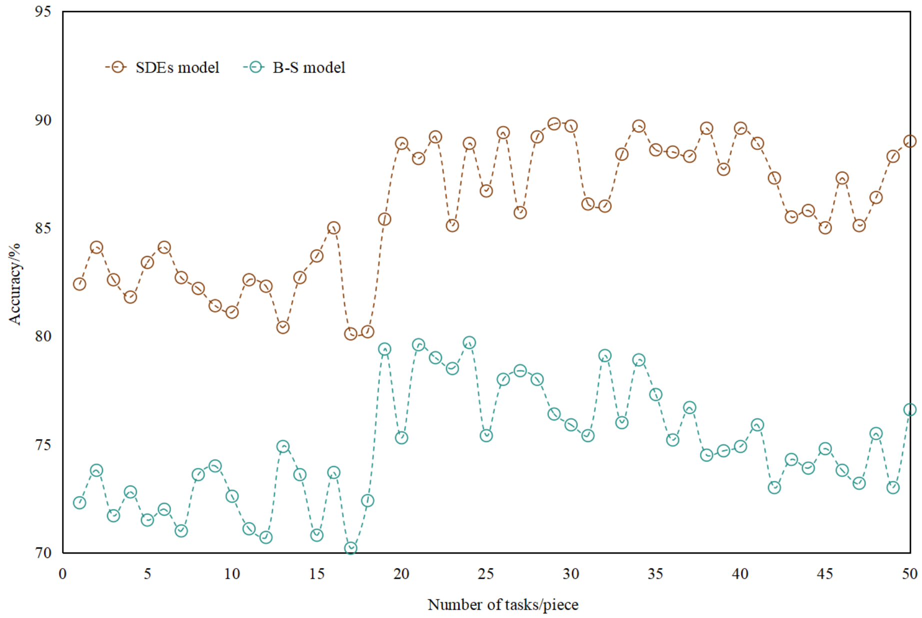

Figure 1: Comparison of prediction accuracy

Figure 2 deeply reveals the significant advantage of the SDEs model constructed in this article in terms of stability, which is particularly outstanding compared to the traditional B-S model. In Figure 2, by simulating the price prediction process under a long time series, it can be observed that the model proposed in this paper has a smoother prediction result and a smaller fluctuation range when dealing with uncertain factors such as market fluctuations and random shocks, demonstrating higher stability. This improvement in stability is mainly due to the fact that the model in this article fully considers the dynamic characteristics and randomness of market prices during the construction process. Through fine model design and parameter optimization, the robustness and reliability of the model are ensured during long-term operation. In addition, this article also introduces advanced numerical solving techniques to effectively reduce numerical errors and divergence phenomena, further enhancing the stability of the model.

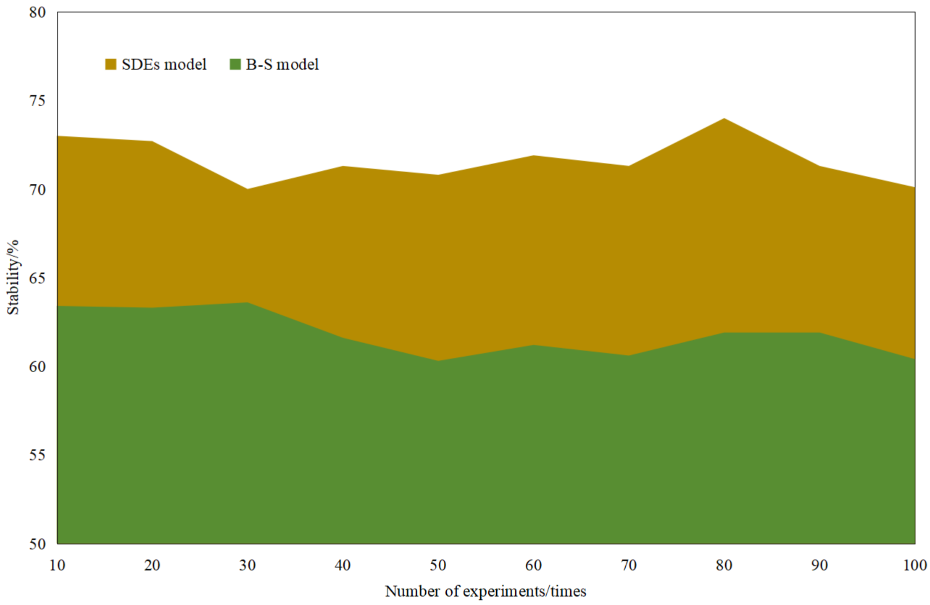

Figure 2: Stability comparison

4. Conclusions

The widespread application of SDEs in the field of financial mathematics is undoubtedly a model of the deep integration of modern financial theory and practice. From classic option pricing problems, to complex interest rate term structure models, to cutting-edge credit risk analysis and investment portfolio optimization strategies, SDEs have become an indispensable tool for financial modeling due to their powerful mathematical expression and accurate capture of real market dynamics. Stability analysis plays a crucial role in these applications. It is not only a key indicator for evaluating the reliability and prediction accuracy of SDEs models, but also the cornerstone for ensuring the sustainable and effective operation of models in complex and changing market environments. A stable SDEs model can maintain the stability and consistency of prediction results over a long period of time, providing reliable decision-making basis for investors and financial institutions. With the continuous development of financial markets and the emergence of financial innovation, we have reason to believe that SDEs and their stability analysis will continue to play an important role in the research and application of financial mathematics in the future, contributing more wisdom and strength to the stable development of financial markets and effective management of financial risks.

References

[1] Lei Ziqi, Zhou Qing. Research on option pricing based on uncertain fractional differential equations under floating interest rates[J]. Journal of Applied Mathematics, 2022, 45 (03): 401-420.
[2] Wang Xinyi, Guo Zhidong. Pricing of Look Back Options Driven by Secondary Diffusion Processes [J]. Journal of Anqing Normal University (Natural Science Edition), 2022, 28 (03): 59-62.
[3] Wang Xinyi, Guo Zhidong. Pricing Model of European Options with Random Interest Rates under Mixed Fractional Brownian Motion Mechanism[J]. Journal of Anqing Normal University (Natural Science Edition), 2022, 28 (01): 92-98.
[4] Gao Shilei, Li Anqi, He Jialu, et al. Numerical simulation and positive comparative analysis of insurance based on CIR model[J]. Science and Technology Innovation Report, 2020,17 (03): 105-108.
[5] Sun Jiaojiao Mellin Transform Method for Pricing European Options under Vasilek Random Interest Rate Model[J]. Economic Mathematics, 2019,36 (03): 21-26.
[6] Zhang Lingxi, Yin Junfeng Numerical Method of KoBol Fractional Order Option Pricing Model[J]. Journal of Tongji University (Natural Science Edition), 2020, 48 (10): 1495-1505.
[7] Hu Yinggang, Jiang Yanqin, Huang Xiaoqian. 7th order weighted compact nonlinear scheme for non-stationary Hamilton Jacobi equation[J]. Journal of Mechanics, 2022, 54 (11): 3203-3214.
[8] Wang Yao, Xue Hong. Pricing of Fragile Options in the Dual Score Vasicek Interest Rate Environment [J]. Computer and Digital Engineering, 2019, 47 (03): 503-507+519.
[9] Wang Jianing, Xue Hong. Pricing of reload options under fractional Brownian motion[J]. Journal of Hangzhou Normal University (Natural Science Edition), 2019, 18 (02): 180-184.
[10] Lan Haifeng, Xiao Feiyan, Zhang Gengen, et al. Error Analysis of Tight Implicit BDF Method for Nonlinear Partial Integral Differential Equations[J]. Journal of Guangxi Normal University (Natural Science Edition), 2020, 38 (04): 82-91.