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Real Chains in Quaternionic Heisenberg Group

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Abstract: *In this paper, we study the real chains in the boundary of quaternionic hyperbolic space. We show that real chain can be decomposed into R-circles and give the uniformly diameter for the finite real chain.*

Keywords: *Quaternionic Hyperbolic Space, Heisenberg Group, R-Chain*

1. Introduction

There is a long history for the study of the discreteness of a group in the hyperbolic geometry. Klein's combination theorem is one of the important tools used to judge the discreteness of the isometric group of the complex hyperbolic space. Kalane and Parker consider the free group generated by two parabolic elements in [1]. They use the Klein's combination theorem to judge the discreteness and construct the fundamental domain by the Cygan spheres. Then Jiang and Xie study the free group generated by unipotent elements in [2].

But in the quaternionic hyperbolic geometry it's still rare examples of constructing the fundamental domains for the discrete groups. In order to discuss this problem in the quaternionic hyperbolic geometry. In this paper we shall discuss the $\mathbb{R}$ -chains in the quaternionic Heisenberg groups as the start point.

The paper arrangement as follows. In section 2, we give some elementary knowledge and some new definitions. In section 3, we shall conclude our main results and proof. In the end, we give the conclusion.

2. Background

We review some elementary contents of the Quaternionic hyperbolic space in this part. For more information refers to [3-5]. We also give some new notions for chains in the quaternionic Heisenberg group and the isometric spheres in terms of geographic coordinates.

2.1 Quaternions Hyperbolic space

2.1.1 Quaternions

A quaternion is a number $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, where $q_i \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$. Its conjugate is given by $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$, and modulus is denoted by $|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. Then $\overline{pq} = \overline{q}\overline{p}$. Let $\Re(q) = (q + \overline{q}) / 2$ be the real part of q and $\Im(q) = (q - \overline{q}) / 2$ be the imaginary part. If the real part is 0, then call it purely imaginary quaternion. If $a = a_1 i + a_2 j + a_3 k$ is a unit quaternion, then any quaternion q has an exponential form:

$$q = |q| e^{a\theta} := |q| (\cos\theta + a\sin\theta). \tag{1}$$

2.1.2 Hyperbolic space

Let $\mathbb{H}^{2,1}$ be the quaternionic vector space with the signature $(2, 1)$. The Hermitian product is given by

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$$\left\langle \hat{z}, \hat{w} \right\rangle = \hat{w}^* J \hat{z} = \overline{w}_3 z_1 + \overline{w}_2 z_2 + \overline{w}_1 z_3, \tag{2}$$

Where $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $\hat{z} = \left(z_1, z_2, z_3\right)^T$, $\hat{w} = \left(w_1, w_2, w_3\right)^T$.

Let $P : \mathbb{H}^{2,1} \setminus \{0\} \to \mathbb{H}P^2$ be the right projection maps onto the quaternion projective space. Then we see that

$$P\left(z_1, z_2, z_3\right)^T = \left(z_1 z_3^{-1}, z_2 z_3^{-1}\right)^T \in \mathbb{H}^2,\ P\left(z_1, 0, 0\right)^T = \infty, P\left(0, 0, z_3\right)^T = o \in \mathbb{H}^2.$$

Suppose that

$$V_0 = \left\{\hat{z} \in \mathbb{H}^{2,1} \setminus \{0\} : \left\langle \hat{z}, \hat{z} \right\rangle = 0\right\},$$

$$V_- = \left\{\hat{z} \in \mathbb{H}^{2,1} \setminus \{0\} : \left\langle \hat{z}, \hat{z} \right\rangle < 0\right\}.$$

Then define the Siegel domain model to be $H^2_{\mathbb{H}} = P(V_-)$ and boundary $\partial H^2_{\mathbb{H}} = P(V_0)$.

The standard lift of a finite point in the quaternionic hyperbolic space is given by:

$$\hat{z} = \begin{pmatrix} z \\ 1 \end{pmatrix} \in P^{-1}\left(z\right), \tag{3}$$

And the lift of the infinite point ∞ is given by $q_\infty = (1, 0, 0)^T$.

2.2 Heisenberg group

The quaternionic Heisenberg group is a group $\mathfrak{N} = \left\{(\zeta, v) : \zeta \in \mathbb{H}, v \in \Im(\mathbb{H})\right\}$ with the multiplication rule

$$(\zeta, v)(w, s) = \left(\zeta + w, v + s + 2\Im(\overline{w}\zeta)\right). \tag{4}$$

One point compactification of quaternionic Heisenberg group identifies the boundary of the quaternionic hyperbolic space. The Cygan-Koranyi norm is given by

$$\left|(\zeta, v)\right|_0 = \left|\left|\zeta\right|^2 + v\right|^{1/2} = \left|\left|\zeta\right|^4 + v^2\right|^{1/4}. \tag{5}$$

Then the Cygan-Koranyi metric can be written by

$$d_{ck}\left((\zeta, v)(w, s)\right) = \left|(\zeta, v)^{-1}(w, s)\right|_0 = \left|\zeta - w, v - s - 2\Im(\overline{w}\zeta)\right|_0. \tag{6}$$

Especially, for any two points x, y in the Heisenberg group the following holds:

$$d_{ck}\left(x, y\right) = \left|\left\langle \hat{x}, \hat{y} \right\rangle\right|^{1/2}. \tag{7}$$

This observation is convenient for the calculation the distance of the points in the Heisenberg group.

Suppose that three points x, y, z are in the Heisenberg group, then define the Cartan's angular invariant to be

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$$\mathrm{Arg}\left(x, y, z\right) = \arccos \frac{\Re\left(-\left\langle \hat{x}, \hat{y}\right\rangle \left\langle \hat{y}, \hat{z}\right\rangle \left\langle \hat{z}, \hat{x}\right\rangle\right)}{\left|\left\langle \hat{x}, \hat{y}\right\rangle \left\langle \hat{y}, \hat{z}\right\rangle \left\langle \hat{z}, \hat{x}\right\rangle\right|}. \tag{8}$$

2.3 Chains

Here we shall give the definition of the circles and quaternionic chains in the quaternionic Heisenberg group.

- $\mathbb{C}$-circle is the intersection with $\partial H^2_{\mathbb{H}}$ of a complex line (or a copy of complex line).
- $\mathbb{R}$-circle is the intersection with $\partial H^2_{\mathbb{H}}$ of a Lagrangian plane (or a copy of Lagrangian plane).
- $\mathbb{H}$-chain is the intersection with $\partial H^2_{\mathbb{H}}$ of a quaternionic line (or a copy of quaternionic line).

2.4 Isometric spheres and real chains

Let $M \in PSp(2,1)$ and $M\left(q_\infty\right) \neq q_\infty$. Then the isometric sphere of M defined by

$$I_M = \left\{x \in \partial H^2_{\mathbb{H}} : \left|\left\langle \hat{x}, q_\infty\right\rangle\right| = \left|\left\langle \hat{x}, M^{-1}(q_\infty)\right\rangle\right|\right\}.$$

If

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \tag{9}$$

Then we have its inverse

$$M^{-1} = \begin{pmatrix} \overline{m}_{33} & \overline{m}_{23} & \overline{m}_{13} \\ \overline{m}_{32} & \overline{m}_{22} & \overline{m}_{12} \\ \overline{m}_{31} & \overline{m}_{21} & \overline{m}_{11} \end{pmatrix}. \tag{10}$$

Note that here $m_{31} \neq 0$. Thus, the isometric sphere I_M has the center $M^{-1}(q_\infty)$ and radius $\rho_{M^{-1}} = \rho_M = 1 / \sqrt{|m_{31}|}$.

In the Heisenberg coordinates, we have

$$M^{-1}(\infty) = \left(2^{-1/2}\overline{m}_{32}\overline{m}_{31}^{-1}, -\Im(m_{33}m_{31}^{-1})\right), \quad M(\infty) = \left(2^{-1/2}m_{23}m_{31}^{-1}, \Im(m_{11}m_{31}^{-1})\right).$$

Let

$$\iota = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

If $(z, t) \neq (0, 0)$ then we have

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$$\iota(z, t) = \left(z\left(-|z|^2 + t\right)^{-1}, \frac{-t}{\left|-|z|^2 + t\right|^2} \right). \tag{11}$$

Let $\alpha \in [0, \pi / 2]$, $\beta \in [0, 2\pi]$, a is a unit purely imaginary quaternion. The isometric sphere with the center o and radius $r > 0$ has the geographical coordinates:

$$s(\alpha, \beta) = \left(r\sqrt{\sin(2\alpha)}e^{a\alpha + a\beta}, ar^2 \cos(2\alpha) \right). \tag{12}$$

The corresponding standard lift is

$$\hat{s}(\alpha, \beta) = \begin{pmatrix} ar^2 e^{a2\alpha} \\ r\sqrt{2\sin(2\alpha)}e^{a\alpha + a\beta} \\ 1 \end{pmatrix}. \tag{13}$$

Now, let $\alpha = \alpha_0$ then $\hat{s}(\alpha, \beta)$ is a quaternionic chain, denoted by $\mathbb{H}$-chain, which has the real dimension 3. If $\beta = \beta_0$ and $\beta = 2\pi - \beta_0$ then $\hat{s}(\alpha, \beta)$ gives a real 3-dimensional $\mathbb{R}$-circle bundle. We define this to be the real chain, denoted by $\mathbb{R}$-chain, in the quaternionic Heisenberg group.

Note that whence a is fixed we have the $\mathbb{H}$-chain and $\mathbb{R}$-chain will be corresponding to a copy of the $\mathbb{C}$-circle and $\mathbb{R}$-circle by the action of $PSp(2, 1)$, respectively. Therefore, we have the following lemma.

Lemma 2.1 $\mathbb{R}$-chain can be decomposed into $\mathbb{R}$-circles.

3. Main results and proof

In this section we shall conclude our main result as follows. We show that a real chain has the uniformly diameter.

Theorem 3.1 *Let R be a finite $\mathbb{R}$-chain in the quaternionic Heisenberg group and $R(a)$ be the a finite $\mathbb{R}$-circle corresponding to the fixed unit purely imaginary quaternion a, fixed by the anti-holomorphic ι_R and $\iota_{R(a)}$ respectively. Let p_θ be a point in $R(a)$ with $\theta \in [0, \pi / 2]$, and $\mathrm{Arg}\left(p_\theta, \iota_R(\infty), \infty\right) = 2\theta - \pi / 2$. Then R has the uniformly bounded maximum diameter*

$$2^{1/2} r \left(\cos^{2/3}(\theta) + \sin^{2/3}(\theta)\right)^{3/4}.$$

Proof: According to [1], we assume that two points $p_{\theta_1}, p_{\theta_2}$ in $R(a)$ can be chosen its sign to have the maximum distance position. Thus, we have

$$d_{ck}(p_{\theta_1}, p_{\theta_2}) = \left| ar^2 e^{2a\theta_2} - ar^2 e^{2a\theta_1} - 2r^2\sqrt{\sin(2\theta_1)\sin(2\theta_2)}e^{a\theta_2 - a\theta_1} \right|_0^{1/2}$$

$$= \left| -2r^2 \sin(\theta_2 + \theta_1)e^{a\theta_2 - a\theta_1} - 2r^2\sqrt{\sin(2\theta_1)\sin(2\theta_2)}e^{a\theta_2 - a\theta_1} \right|_0^{1/2}$$

$$= \sqrt{2}r\left(\sin(\theta_2 + \theta_1) + \sqrt{\sin(2\theta_1)\sin(2\theta_2)}\right)^{1/2}$$

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$$= \sqrt{2}r\left|\sqrt{\sin(\theta_1)\cos(\theta_2)} + \sqrt{\cos(\theta_1)\sin(\theta_2)}\right|. \tag{14}$$

Then by the Lemma 3.1 of [1], we see that

$$\sqrt{\sin(\theta_1)\cos(\theta_2)} + \sqrt{\cos(\theta_1)\sin(\theta_2)} \le \left(\cos^{2/3}(\theta_2) + \sin^{2/3}(\theta_2)\right)^{3/4}. \tag{15}$$

It follows that the maximum distance is

$$d_{max}(\theta_2, R) = 2^{1/2}r\left(\cos^{2/3}(\theta_2) + \sin^{2/3}(\theta_2)\right)^{3/4}. \tag{16}$$

At last, by lemma 2.1 we obtain the uniformly diameter of R:

$$2^{1/2}r\left(\cos^{2/3}(\theta_2) + \sin^{2/3}(\theta_2)\right)^{3/4}.$$

4. conclusion

In this paper, we give the notion of chain in the quaternionic Heisenberg group then show that the $\mathbb{R}$-chains can be decomposed into $\mathbb{R}$-circles. In the end, we give the diameter for the $\mathbb{R}$-chain.

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