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The Proof of The Four Color Conjecture Based on Graph Theory Logic

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Abstract: *Since 1850, the four-color theorem has become one of the most controversial problems in the history of mathematics and one of the "three major problems in modern mathematics in the world" because of its "intuitive simplicity but difficult to prove". In this paper, we use the graph theory to prove the four-color theorem by the unified construction, decomposition and regular splicing of the planar graph. Firstly, based on the theory of maximal planar graph, this paper decomposes the maximal planar graph through the graph surrounding coloring model, and constructs the adjacency relationship. This process not only preserves the original adjacency property of planar graphs, but also provides a basis for the subsequent coloring relations. Based on these decompositions, the four-colorability of the graph surrounding coloring model is further proved by using the number line abstract relation that can be logically understood. Then, according to the adjacency relationship of the surrounding shading models, the three-dimensional number axis shading model is constructed, and the adjacency relationship is simplified. This simplification not only reduces the complexity of coloring combinations, but also makes the analysis of adjacency coloring relations more intuitive. Based on the simplified coloring combination, this paper re-colors the odd and even cycles surrounding the adjacency of the coloring model, and further simplifies the combination of triangular adjacency coloring relations through the equivalent substitution of the values of the adjacent endpoints on the three-dimensional number axis. On the basis of the above simplified combination, through a series of proof steps, we verify the four-colorability of planar graphs under various combinations. These steps include, but are not limited to, an extension of the adjacency distance around the module based on the conclusions demonstrated above, an extension of the adjacency case outside the triangular adjacency window, and an extension by a regular combination of odd and even adjacency end cycles (OAEC and EAEC). By these extensions, the conclusion is proved to cover the whole planar graph. Finally, this paper proposes a logical proof method for the four-color theorem, which simplifies the complexity of the coloring relation through a series of graph transformation rules, and extends the rules effectively, and finally provides a proof of the four-color theorem for finite planar graphs. This method not only shows the potential of graph theory logic in solving complex mathematical problems, but also provides a new perspective and tool for the corresponding field research.*

Keywords: *Plane graph decomposition construction; Surround coloring model; Number axis abstract coloring relationship; Graph modeling and simplification of adjacency relationships of each surround model; The proof of four-colorability for adjacency in each surrounding module; Planar Graph Assembly*

1. Introduction

In the vast and luminous history of mathematics, the four-color conjecture, first proposed by Francis Guthrie in 1852[1], has stood out as one of the most celebrated unsolved puzzles in mathematical history[2] with its elegant formulation and profound logical abstraction. It has functioned akin to a towering mathematical peak and a guiding beacon, drawing the attention and explorative efforts of innumerable mathematicians[3]. Precisely because of its simplicity in statement and its profound mathematical essence, it has inspired the expansion of mathematical theory and the innovation of proof methods, acting as a bridge that connects classical mathematics with modern mathematics[4].The four-color conjecture states that any map on a plane, when divided into contiguous regions, can be colored with merely four colors to ensure that no adjacent regions share the same color[2]. This seemingly simple mathematical proposition has, over nearly two centuries, witnessed the evolution of mathematical methodologies and the deep integration of mathematics with fields such as computer science and information theory[5].

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The earliest known written record was found in a book review penned by De Morgan on April 14, 1860[6]. The problem gained significant attention in the mathematical community when British mathematician Arthur Cayley (1821-1895) raised the question on June 13, 1878, at a meeting of the London Mathematical Society, inquiring whether the four-color problem had been proven[2]. This question sparked considerable interest, leading to a continuous stream of purported proofs being submitted to major mathematical journals and centers around the world.In 1879, Alfred Bray Kempe (1849-1922), a mathematics graduate from Trinity College, Cambridge, offered the first purported "proof"[7]. Eleven years later, in 1890, Percy John Heawood (1861-1955) identified a counterexample—now known as the Heawood graph—which revealed a flaw in Kempe's argument[8], once again transforming the four-color problem into an unsolved mystery.

In 1976, American mathematicians Kenneth Appel and Wolfgang Haken announced that they had proven the four-color theorem with the assistance of two electronic computers, which undertook 10 billion logical operations over 1200 hours, effectively solving the four-color problem that had persisted for 124 years[9][10]. Following the first proof, mathematicians did not abandon their search for a rigorous mathematical proof. In the 1990s, the team led by Paul Seymour, who had been dedicated to studying the four-color problem, claimed to have found a more simplified proof method. This proof relied on computers for a streamlined process, reducing the required machine time from 1200 hours to just 24 hours. The second-generation proof met the requirements for manual verification[11].With the development of computer science, a so-called formal proof emerged in the mathematical community. The idea behind this method is to write a code that not only describes what a machine should do but also characterizes why it must perform in that manner. The validity of the proof is an objective mathematical fact checked by various programs, while the correctness of the programs themselves can be determined experimentally by running them with multiple inputs. Despite numerous challenges, the four-color theorem received its third-generation proof. In 2005, Georges Gonthier published his formal proof of the four-color theorem, which could be fully verified by the Coq proof assistant system[12][13].However, for mathematicians, a purely mathematical proof, and preferably a simpler one, remains the most convincing method. Thus, some fundamentally opposed the use of computers in proofs[14][15]. To date, numerous graph theorists continue to seek a concise and pure mathematical proof.

Graph Theory, serving as a cornerstone in the study of discrete mathematical structures, has provided a robust mathematical framework and analytical tools for solving the four-color conjecture[16]. Since the early 20th century, Graph Theory has emerged as an independent discipline, not only enriching mathematical theory but also profoundly impacting research in fields such as computer science, information theory, and network analysis[17-19]. The coloring problem of planar graphs in Graph Theory, closely tied to the four-color conjecture, has become a central focus in studies of the conjecture[20]. Simultaneously, the proof of the four-color conjecture has propelled the advancement of Graph Theory as a discipline[21][22].The aforementioned computer-assisted proofs reflect the human mind's underdeveloped logical reasoning regarding this problem, specifically a lack of thorough analysis of the conditions for the theorem's validity. If one persists in conducting a deeper analysis of the conditions under which the four-color conjecture holds true, using the perspective and logical framework of Graph Theory, and applying multi-dimensional thinking on this foundation, there still exists potential for a very concise method to prove the validity of the four-color theorem.

In this article, I began by deeply analyzing the coloring problems of planar graphs, considering their logical intricacies. To facilitate a unified analysis and handling of coloring relationships in planar graphs, without disrupting the adjacency relationship, I transformed the planar graph into a maximal planar graph by adding adjacency edges internally [23-24]. On this foundation, I then systematically decomposed the planar graph based on the ordering of the degree centrality in the graph wrapping coloring model [25], proving its four-colorability through the application of abstraction of numerical axis relationships. Subsequently, I established a unified triangle adjacency relationship among all the wrapping models and abstracted it into a three-dimensional numerical axis adjacency model. Simultaneously, by categorizing the endpoints of adjacency in the triangle adjacency zone into equivalent classes based on their values on the numerical axis, I simplified the combination relationships of adjacency coloring. Under this simplification, and leveraging the proof of the four-colorability of the surrounding coloring models in the previous plane graphs, I re-examined the adjacency circles of the surrounding coloring models, proving that in cases where the odd circles require only one more color than the even circles to maintain coloring within the circles, without affecting the coloring within the circles. I then used equivalent substitution rules for insertion and replacement to further simplify the combination of coloring on the endpoints of adjacency in each surrounding circle. Finally, based on the above simplification of the combination, I proved the four-colorability of various combinations of planar graphs. By reversing the construction based on the proof conclusions and simplified combination graphics, I step-by-step expanded the proof conclusions. Through the merging of pairs of adjacent endpoints, I expanded the situations regarding the center adjacency distance of the surrounding modules (e.g., from d_1=3 to d_1=2). By constructing the triangular adjacencies of each surrounding module, I proved various combinations of adjacency outside the adjacency windows. Through the rules of combination between odd-numbered (OAEC) and even-numbered (EAEC) edge-point circles, I constructed the entire finite plane graph, proving that by covering the entire finite plane graph, the four-colorability of the planar graph is thus established.

2. Planar Graph Coloring Model Construction

2.1. Terminology Definition

Definition of a planar graph: If a graph G is a planar graph, it indicates that the edges representing the adjacency of endpoints intersect only at the endpoints.

Definition of finite boundary maximal planar graph: Let G be a finite planar graph, if for any two non-adjacent points u, v in the interior of G, after adding the edge (u, v), it is not a planar graph, then G is called a maximal planar graph in the boundary of a finite planar graph.

Definition of degree: In a planar graph G, the number of edges associated with a vertex v is called the degree, simply denoted as deg v.

Definition of connectivity: In a planar graph, after the coloring model decomposition of the surrounding modules is completed, if an surrounding module shares an edge with the remaining adjacent surrounding modules in the decomposition, we may color the center of this residual adjacent surrounding module with the same color, defining the adjacent surrounding modules as connected. This is a special constraint definition for this article[26].

Definition of odd and even cycles: Let G be a graph precisely composed of a cycle containing n vertices. When n is an odd number, we refer to C_n as an odd cycle; when n is an even number, we refer to C_n as an even cycle.

Definition of the surrounding coloring model: In the process of coloring endpoints in a planar graph, focusing on a specific vertex (with deg v ≥ 3) as the central vertex, adjacent vertices form an surrounding relationship with this central vertex as the core. The coloring model is applied in a graph where the adjacent vertices, besides the surrounding relationship, also have direct adjacency relationships with each other. This definition excludes virtual connection edges and their endpoints.

Definition of the encircling cycle: In the encircling coloring model, the encircling cycle is defined as the outermost ring of vertices that surrounds the central vertex, forming a color cycle. On this color cycle, vertices can establish adjacency relationships with vertices located on adjacent encircling cycles.

Definition of the combined surrounding module: In this context, the combined surrounding module refers to the assembly of surrounding modules produced through localized connectivity construction of surrounding modules. In a combined surrounding module, each encircling center is colored with the same color.

2.2. Coloring Relationships and Conditions in Planar Graphs

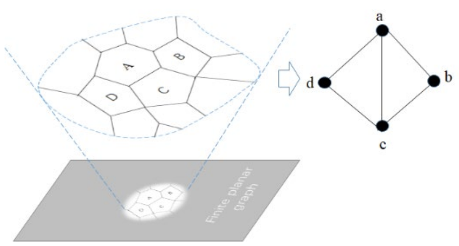

Figure 1: Modeling the Coloring of Planar Graphs

The four-color problem in planar graphs: This can be simplified by viewing each region as a single point. If any two regions are adjacent (or not adjacent), the points representing these two regions are connected by a line (otherwise, they are not connected), thereby establishing a graph coloring model. Consequently, the problem of whether a planar graph can be colored with four colors transforms into the question of whether the endpoints in the graph model can be colored with four colors. As illustrated in Figure 1, regions A, B, C, and D are respectively simplified into the endpoints a, b, c, and d within the graph model. The adjacency relationships between regions A, B, C, and D are simplified into the undirected edges represented as ab, bc, cd, ad, and ac. Consequently, the problem of coloring regions in the planar graph evolves into an issue of coloring the endpoints in the graph model, with the stipulation that endpoints connected by an adjacent edge must be colored differently.

2.3. Constructing a Maximal Planar Graph

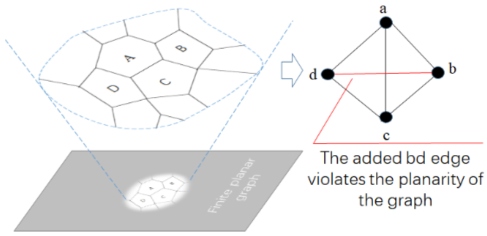

Figure 2: Construction of interior maximal planar graph of finite planar graph

Based on the above established coloring model of finite planar graph, if any two non-adjacent points in it are added with a connecting edge, and then the added edge is not a planar graph, then it is said that a maximal planar graph is constructed in a finite planar graph. As shown in Fig. 2, a maximum planar graph inside a finite planar graph is constructed, and the endpoints a, b, c and d inside the planar graph have formed a triangular adjacency. If an additional bd edge is added inside the planar graph, it will intersect with the ac edge, thus destroying the characteristics of the planar graph. That is to say, on the basis that the adjacency relation of each end inside the planar graph is triangular adjacency, adding any connecting edge inside the planar graph will destroy the characteristics of the planar graph. In order to unify the planar graph coloring model, this paper adds a connecting edge to each connectable end point in a planar graph, which is converted into a triangular adjacency relationship, and then constructs a maximal planar graph in a finite planar graph.

2.4. Decomposition of Planar Graphs and Simplification of Coloring Diagrams

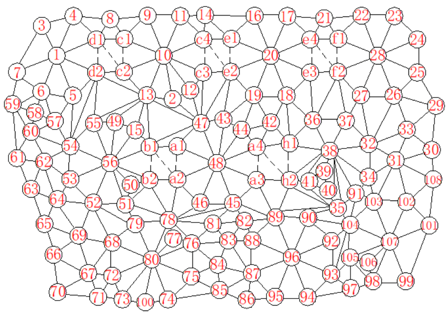

Figure 3: Interior maximal planar graph

Without loss of generality, through the above processing, a maximal planar graph is established inside the graph to be colored, as shown in Figure 3. The number of vertices in the figure is limited to N, and the dotted lines connecting the endpoints in alphabetical order indicate the number of hidden endpoints, for example, the dotted lines connecting the endpoints a1 and a2 in the figure indicate that even or odd endpoints are omitted; The solid line connection of the letter and Arabic numeral mark end point pair and the Arabic numeral and Arabic numeral mark end point pair indicates the adjacent relationship. For example, in the figure, the solid line connection of the a1 end point and the 48 end point and that of the 47 end point and the 48 end point indicate that the end points are adjacent. In this way, each end point of that plan view is uniformly mar and each mark is unique. Based on the construction of maximal planar graphs and the characteristics of the degree of endpoints, a finite planar graph can always calculate the degree of each endpoint and can be sorted from large to small according to the degree. As shown in Fig. 4, the degree of each endpoint of the internal maximum planar graph is calculated, so that the corresponding relationship between each endpoint of the planar graph and the degree thereof is constructed, and a planar graph endpoint degree corresponding table is generated as shown in Table 1, wherein the relationship between the degrees of letters in Table 1 is $n > m > q > p > j > i > h > f > 10$.

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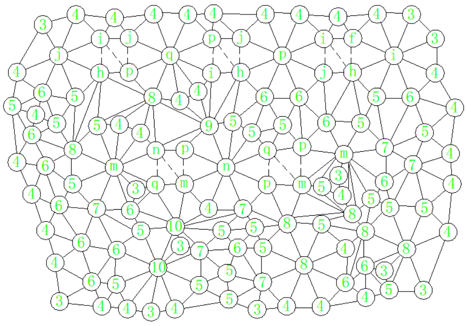

Figure 4: Calculation of degree of each end point of interior maximal planar graph

Table 1: Correspondence table of node degrees in the plane graph

endpoint	deg v	endpoint	deg v	endpoint	deg v	endpoint	deg v
1	j	c4	p	28	i	b1	n
…	…	…	…	…	…	b2	q
d1	i	e1	j	38	m	…	…
d2	h	e2	h	…	…	48	n
c1	j	e3	j	h1	p	…	…
c2	p	e4	i	h2	m	56	m
…	…	f1	f	a1	p	…	…
10	q	f2	h	a2	m	78	10
…	…	20	p	a3	p	…	…
c3	i	…	…	a4	q	108	4

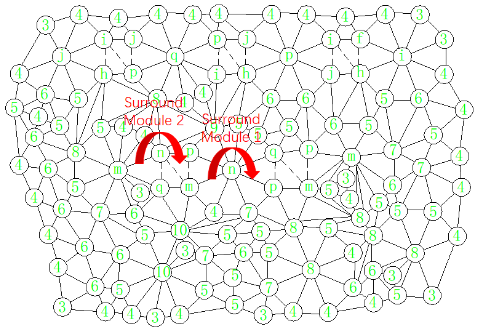

Figure 5: Construction of adjacency modules for vertices with degree n

Sorting the vertices by degree according to Table 1, it is always possible to construct graph adjacency modules centered around each vertex, as demonstrated in Figure 5 with the construction of adjacency modules for vertices of degree n. The figure illustrates two instances of adjacency modules for vertices of degree n, specifically, module 2 centered around vertex b1, and module 1 centered around vertex 48.In order to standardize the decomposition and construction of adjacency modules across the planar graph, it is stipulated that the process should commence from the center of the graph. Subsequent module decompositions prioritize starting from the closest and most central vertex relative to the adjacency module decomposed in the previous step. The centrality of vertices is calculated based on a unified directional coordinate system for the planar graph, using the shortest graph edge distances to the top (e1), bottom (e2), left (e3), and right (e4) from a central vertex, with the centrality ratio (γ) serving as the criterion. A smaller γ indicates a more central position.

$$\gamma = 2 \times \left(\frac{\mathrm{I}\ e1 - e2\ \mathrm{I}}{e1 + e2} + \frac{\mathrm{I}\ e3 - e4\ \mathrm{I}}{e3 + e4}\right) \times 100\%$$

In Figure 5, the edge distances for surrounding module 1, centered around endpoint 48, are (left 6 (e3), right 5 (e4), top 4 (e1), bottom 5 (e2)), as depicted in Figure 6, the edge distance diagram of the central endpoint for surrounding module 1, with γ_1 calculated at 40.4%.

$$\gamma_1 = 2 \times \left(\frac{\mathrm{I}\ 4 - 5\ \mathrm{I}}{4 + 5} + \frac{\mathrm{I}\ 6 - 5\ \mathrm{I}}{6 + 5}\right) \times 100\% = 40.4\%$$

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In Figure 5, the edge distances for surrounding module 2, centered around endpoint b1, are (left 4 (e3), right 9 (e4), top 3 (e1), bottom 6 (e2)), as illustrated in Figure 7: the edge distance diagram of the central endpoint for surrounding module 2, with γ_2 calculated at 143.6%.

$$\gamma_2 = 2 \times \left(\frac{|4-9|}{4+9} + \frac{|3-6|}{3+6} \right) \times 100\% = 143.6\%$$

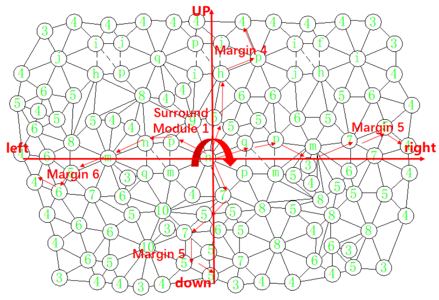

Figure 6: Illustrates the edge distance diagram of the central endpoint for surround module 1

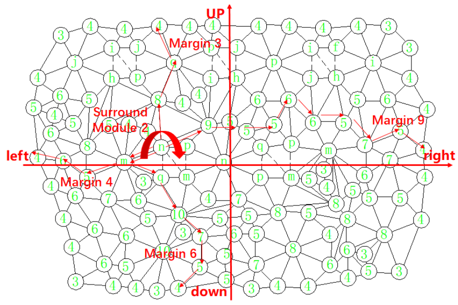

Figure 7: Illustrates the edge distance diagram of the central endpoint for surround module 2

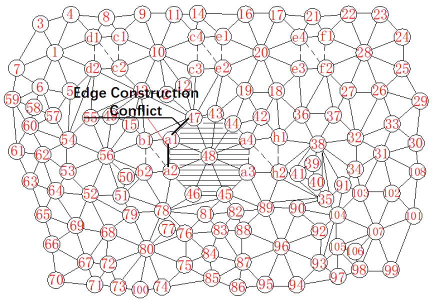

Figure 8: Decomposition and Construction of adjacency modules for a center vertex with deg v=n

Given that γ_1 is less than γ_2, adjacency module 1 centered around vertex 48 is relatively central and thus prioritized for decomposition and construction. As shown in Figure 8: Decomposition and construction of adjacency modules for a center vertex with deg v=n. Since there is a conflict between edges (a1,a2) and (a1,47) of adjacency module 2 centered around vertex b1, and adjacency module 1 centered around vertex 48, it cannot proceed with further decomposition and construction.

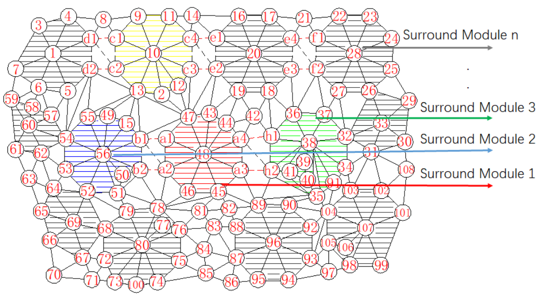

Figure 9: Simplification of planar graph decomposition

Continue to decompose and construct according to the above rules of decomposition and construction

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of planar graphs. A finite planar graph can always be decomposed into different sub-graphs of surrounding modules, as shown in Figure 9. Fig. 9 extends the adjacent edges (a1, b1), (a2, b2), (a4, h1), (a3, h2), and (c1, d1) to be connected by dotted lines on the basis of fig. 3, and indicates that even or odd end points are omitted in the adjacent edges. Therefore, the planar graph can be decomposed into k surrounding modules without loss of generality. In Fig. 9, the configurations of the surrounding modules 1 (red surrounding module), 2 (blue surrounding module), and 3 (green surrounding module) around the central end point 48, 56, and 38 are illustrated in an exploded plan view. There are other configurations of the surrounding modules around the central end point in the figure, which are not labeled one by one to simplify the discussion.

2.5. Construction of the Planar Graph Coloring Model

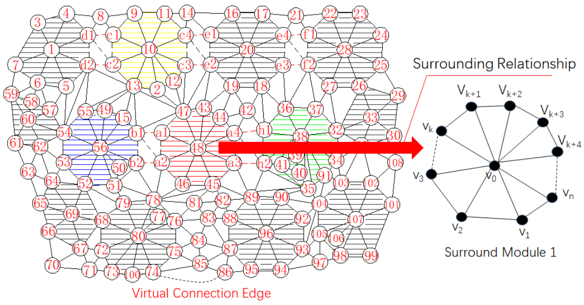

Figure 10: Subgraph of the surrounding module 1 for the graph decomposition

Based on the unified decomposition mode of surrounding modules around the center and end points, the planar graph shading model can be simplified into a combination of surrounding modules. To prove that planar graphs are four-colorable, we first consider the surrounding modular subgraphs. For example, in the sub-graph of the surrounding module 1 of the planar graph decomposition of Fig. 10, the center endpoint 48 is replaced by v_0, the endpoint 48 adjacent to 45 is replaced by v_1, the endpoint 48 adjacent to 46 is replaced by v_2, and so on. The sub-graph coloring can be regarded as the surrounding module coloring problem around the center endpoint v_0. For the convenience of the general proof of the problem, take the full surrounding model around the endpoint v_0. In the figure, the adjacent end points around the end point v_0 are connected end to end to form a full surround model. Based on the full surround relationship, there must be a surrounding circle, and the surrounding circle C_n is v_1, v_2, v_3, v_k, v_{k+1}, v_{k+2}, V_{k+3}, V_{k+4}, v_n, v_1.

If the surrounding edges adjacent to a central endpoint are not all present, as illustrated in Figure 10, if we attempt to construct a complete surrounding model around the central endpoint 85, there is a missing connection edge between endpoints 74 and 86, which prevents the formation of a fully enclosed circumferential loop. Virtual endpoint connection edges can be introduced to complete the circumferential relationship, without constraining their adjacency, and this does not affect the decomposition construction of the planar graph while satisfying the unified construction of the full surrounding model for planar graphs. Therefore, the four-colorable proof of surrounding module 1 constructed by planar graph decomposition can be applied to various coloring proofs in the case that the number of endpoints surrounding the central endpoint v_0 in graph G is n.

3. Proving the Four-Colorability of Graph Surrounding Modules Based on Graph Theory Logic

3.1. Four-coloring proof for the surrounding module 1 based on the decomposition of a planar graph

3.1.1. Establishment of Plane Graph Coloring Relationships

Because the planar graph coloring model can be simplified as a graph model surrounding the endpoint v_0, the adjacency relations of the endpoints are not only those on the circle, such as v_1 and v_2 adjacency, v_1 and v_n adjacency, but also those across the circle, such as v_1 and v_3 adjacency, v_1 and v_{n-1} adjacency; The essence is relationship mapping, so the number axis relationship model can be established from the mathematical logic, such as the number axis relationship model of the planar graph coloring in Fig. 11. The number axis 1 is the adjacent relationship on each end point circle, and the number axis 2 is the value domain of each end point cross-circle adjacent relationship.

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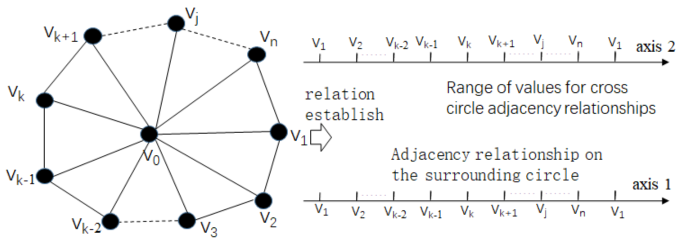

Figure 11: Numerical axis relationship model for coloring of planar graphs

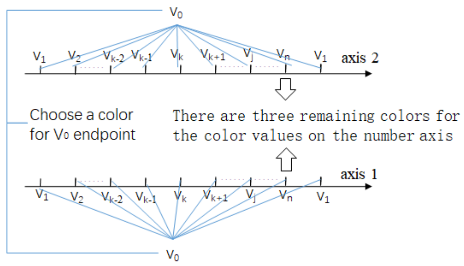

Figure 12: Mapping relationship of numerical axis values for planar graph coloring

The circle surrounding the loop is unfolded onto a numerical axis, maintaining an equivalent mapping between the axis and the circle, such that the adjacency relationships on the planar graph's surrounding loop are preserved when mapped onto numerical axis 1. As illustrated in the mapping relationship of the planar graph coloring numerical axis in Figure 12, the adjacency relationships across the circle are mapped onto an equivalent numerical axis 2. In addressing the four-colorability problem of the planar graph, since all surrounding terminals are adjacent to terminal v_0, terminal v_0 must be colored with a different color (red R). On the numerical axis, the terminals can only be colored with the remaining three colors (blue L, green G, yellow Y).

3.1.2. Four-Colorability Proof Based on Plane Graph Coloring Relationships

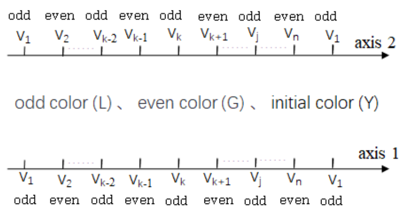

Figure 13: Number axis endpoint value model of planar graph coloring

Combining the plane graph number axis coloring model, suppose the remaining endpoints are colored as odd color (L), even color (G), and initial color (Y). As shown in Figure 13, the plane graph coloring number axis endpoint value model, for all endpoints on the number axis, odd and even values alternate, satisfying the value relationship of the number axis and the adjacent relationship of the endpoints on the surrounding loop. In this case, the four-colorability argument for the plane graph can be divided into three scenarios.

3.1.2.1. Proving the conclusion of four colorability for any arbitrary endpoints on number axis 1 adjacent to any arbitrary endpoints on number axis 2 in a plane diagram.

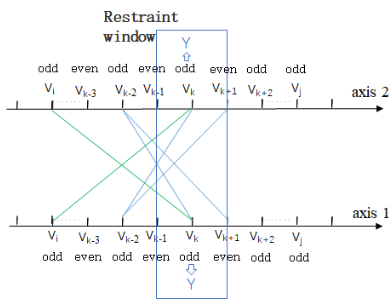

Figure 14: Plane graph coloring number axis substitution and constraint relationship

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Since any endpoint on the number axis takes only one of odd or even, defining the endpoint value is without loss of generality. As shown in Figure 14, the plane graph coloring number axis value substitution and constraint relationship, take the endpoint v_{k-2}, its adjacent edge with the endpoint v_k is a blue edge; other endpoints far away from v_{k-i} (i≥2) can be taken, such as the endpoint v_i, its adjacent edge with the endpoint v_k is a green edge, and its endpoint v_i value does not affect the constraint window number axis odd and even replacement relationship. The v_{k-2} endpoint is odd on the number axis 1, and its adjacency with the number axis 2 is either (odd, even) adjacent or (odd, odd) adjacent; if it is (odd, even) adjacent, it meets the plane graph coloring condition, and no number axis value replacement processing is done, such as the v_{k-2} endpoint is (odd, even) adjacent to the v_{k+1} endpoint; if it is (odd, odd) adjacent, it does not meet the plane graph coloring condition, then the corresponding adjacent endpoint number axis value is replaced with the initial color (value is Y), such as the v_{k-2} endpoint is (odd, odd) adjacent to the v_k endpoint, and the number axis v_k endpoint value is odd replaced with the initial color Y. The initial color replacement on the number axis will not change the original number axis odd and even color value relationship, but it needs to consider the adjacency value relationship of the adjacent endpoints, that is, the v_{k-1} endpoint and v_{k+1} endpoint adjacent to the v_k endpoint (since v_k is odd, they are both even) whether there is (even, even) adjacency with other endpoints, and the initial color substitution is also needed, then the constraint window is the v_{k-1} endpoint to the v_{k+1} endpoint. In the constraint window, if there is a demand for initial color substitution, it contradicts the initial color substitution of the adjacent endpoint v_k, then the plane graph cannot be four-colored, if not, then the plane graph can be four-colored.

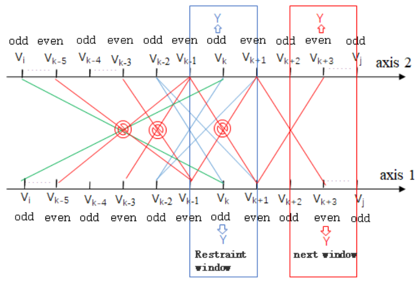

Figure 15: The numerical substitution relationship in the value constraint window of the color number axis in planar graph

Due to the characteristics of the plane graph, the v_{k+1} endpoint cannot be adjacent to the v_{k-1} endpoint, the v_{k-1} endpoint cannot be adjacent to the v_{k-3} endpoint, and the v_{k-1} endpoint cannot be adjacent to the v_{k-5} endpoint. As shown in Figure 15, the red edges $v_{k-1}v_{k+1}$, $v_{k-3}v_{k-1}$, and $v_{k-5}v_{k-1}$, which are adjacent to the v_{k+1} endpoint and v_{k-1} endpoint, v_{k-3} endpoint and v_{k-1} endpoint, and v_{k-5} endpoint and v_{k-1} endpoint in the plane graph coloring number axis value constraint window value substitution relationship, respectively, intersect with the assumed blue edge $v_{k-2}v_k$ adjacent to the v_{k-2} endpoint and v_k endpoint. The red adjacent edges do not satisfy the characteristics of the plane graph, thus cannot form an adjacency relationship. Within the constraint window, the v_{k+1} endpoint can only be (even, even) adjacent to v_{k+3} (including v_{k+3}), v_{k+3} (the number axis value is even replaced with the initial color Y) is not in the constraint window, enters the next constraint window, and is isolated from this constraint window, so the v_{k-1} endpoint and v_{k+1} endpoint adjacent to the v_k endpoint have no (even, even) adjacent initial color substitution demand with other endpoints within the constraint window, then the plane graph can be four-colored. Similarly, without loss of generality, for v_{k-2} taken on the number axis 1, axis 2 as even, only the corresponding value replacement needs to be done, which also satisfies the above proof; and the above proof constraint window relationship satisfies the periodic function characteristics in value, therefore, the adjacent coloring relationship replacement on the overall number axis satisfies the plane graph can be four-colored.

3.1.2.2. The proof of four-colorability due to the circumferential relationship on the number axes 1 and 2, causing the starting endpoint and terminating endpoint to be colored differently on the plane graph

Due to the value relationship of the loop on the number axis, the final endpoint and the starting endpoint will have the same odd and even values. In this case, only the initial color Y replacement needs to be done for the final endpoint, which does not change the overall odd and even value relationship of the number axis. Without loss of generality, as shown in Figure 16, the constraint relationship of the starting point and endpoint values of the plane graph coloring number axis 1, the starting endpoint v_1 and the endpoint v_n are (odd, odd) adjacent, then the initial color Y is replaced for the endpoint v_n. The initial color replacement on the number axis will not change the original odd and even value relationship of the number axis, but it needs to consider the adjacency value relationship of the adjacent endpoints, that is, the v_{n-1} endpoint and v_1 endpoint adjacent to the v_n endpoint (since the original value of v_n is odd, then the v_1 endpoint is odd, and the v_{n-1} endpoint is even,) whether there is (even, even), (odd, odd) adjacency with other endpoints, and the initial color substitution is also needed, then the constraint window is the v_1 endpoint to the v_{n-1} endpoint. In the constraint window, if there is a demand for initial color substitution, it contradicts the initial color substitution of the adjacent endpoint v_n, then the plane graph cannot be four-colored, if not, then the plane graph can be four-colored. Under this condition, two cases are discussed.

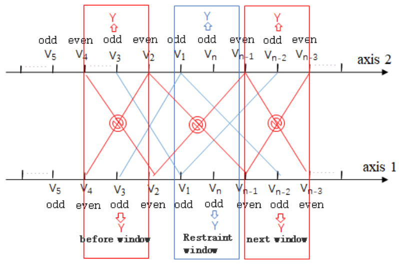

Figure 16: The value constraint relationship between the starting and ending points on the color number axis in planar graph 1

Case1,if the v_1 endpoint is adjacent to the v_{n-2} endpoint, due to the characteristics of the plane graph, the v_{n-1} endpoint cannot be adjacent to the v_{n-3} endpoint and the v_2 endpoint; if the v_1 endpoint is adjacent to the v_3 endpoint, due to the characteristics of the plane graph, the v_2 endpoint cannot be adjacent to the v_{n-1} endpoint and the v_4 endpoint. As shown in Figure 16, the red edges $v_{n-1}v_{n-3}$ and $v_{n-1}v_2$, which are adjacent to the v_{n-1} endpoint and v_{n-3} endpoint, and v_{n-1} endpoint and v_2 endpoint in the plane graph coloring number axis starting point and endpoint value constraint relationship 1, respectively, intersect with the assumed blue edge v_1v_{n-2} adjacent to the v_1 endpoint and v_{n-2} endpoint. The red adjacent edges do not satisfy the characteristics of the plane graph, thus cannot form an adjacency relationship. Within the constraint window, the v_1 endpoint is (odd, odd) adjacent to the v_{n-2} endpoint and v_3 endpoint, but there is no need for the v_1 endpoint to be replaced with the initial color Y, the corresponding v_{n-2} endpoint and v_3 endpoint are replaced with the initial color Y in the next constraint window and the previous constraint window, respectively, and within the next constraint window and the previous constraint window, due to the characteristics of the plane graph, there is no (even, even) adjacency relationship for the adjacent terminals. Therefore, the v_{n-1} endpoint and v_1 endpoint adjacent to the v_n endpoint have no adjacent initial color substitution demand with other endpoints within the constraint window, then the plane graph can be four-colored.

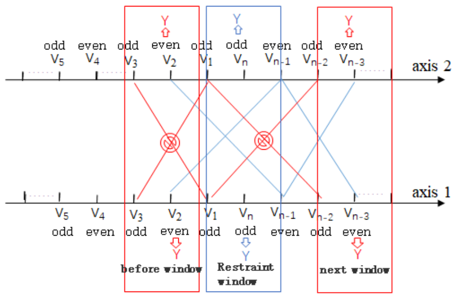

Figure 17: Plane graph coloring number axis starting point and endpoint value constraint relationship 2

Case2, if the v_{n-1} endpoint is adjacent to the v_{n-3} endpoint, due to the characteristics of the plane graph, the v_{n-2} endpoint cannot be adjacent to the v_1 endpoint; if the v_{n-1} endpoint is adjacent to the v_2 endpoint, due to the characteristics of the plane graph, the v_1 endpoint cannot be adjacent to the v_3 endpoint. As shown in Figure 17, the red edge $v_{n-2}v_1$ adjacent to the v_{n-2} endpoint and v_1 endpoint in the plane graph coloring number axis starting point and endpoint value constraint relationship 2 intersects with the assumed blue edge $v_{n-1}v_{n-3}$ adjacent to the v_{n-1} endpoint and v_{n-3} endpoint; the red edge v_1v_3 adjacent to the v_1 endpoint and v_3 endpoint intersects with the assumed blue edge $v_{n-1}v_2$ adjacent to the v_{n-1} endpoint and v_2 endpoint; the red adjacent edges do not satisfy the characteristics of the plane graph, thus cannot form an adjacency relationship. Within the constraint window, the v_{n-1} endpoint is (even, even) adjacent to the v_{n-3} endpoint and v_2 endpoint, but there is no need for the v_{n-1} endpoint to be replaced with the initial color Y, the corresponding v_{n-3} endpoint and v_2 endpoint are replaced with the initial color Y in the next constraint window and the previous constraint window, respectively, and within the next constraint

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window and the previous constraint window, due to the characteristics of the plane graph, there is no (odd, odd) adjacency relationship for the adjacent terminals. Therefore, the v_{n-1} endpoint and v_1 endpoint adjacent to the v_n endpoint have no adjacent initial color substitution demand with other endpoints within the constraint window, then the plane graph can be four-colored.

Similarly, the endpoint v_n and the endpoint v_1 are both taken as even on the number axes 1 and axes 2, only the corresponding value replacement needs to be done, which also satisfies the above proof; and the above proof constraint window relationship satisfies the periodic function characteristics in value, therefore, the adjacent relationship replacement on the number axis satisfies the plane graph can be four-colored.

3.1.2.3. Combination of Cases 1 and Cases 2

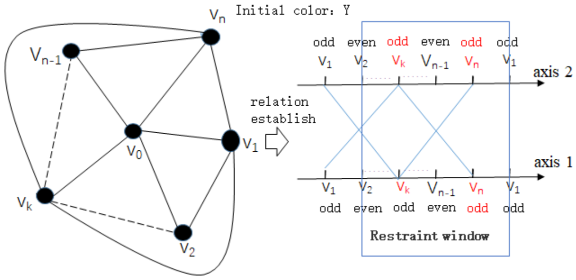

Figure 18: Plane graph coloring model for combination of cases 1 and cases 2

In response to the combination of any endpoint on the number axis 1 crossing the loop to any endpoint on the number axis 2 and the circumferential relationship on the number axes 1 and axes 2 causing the starting endpoint and terminating endpoint to be adjacent, leading to the need for initial color substitution for the plane graph to be four-colored, without loss of generality, a schematic diagram and number axis adjacency model are established, as shown in Figure 18, the plane graph coloring cases 1 and cases 2 combination diagram model.

In the diagram model, in addition to circling around the endpoint v_0, the endpoints v_1, v_2, ,,,v_k,,,,v_{n-1}, v_n also have adjacency relationships between v_1 and v_k, and v_k and v_n. The above combination of odd circling and cross-loop adjacency on the loop will inevitably lead to the situation of initial color combination substitution within the constraint window, resulting in the conflict of initial color substitution at the beginning and end. For example, in the diagram, the adjacency conflict between endpoints v_1 and v_k requires the replacement of the initial color at endpoint v_k, and the adjacency between endpoints v_1 and v_n requires the replacement of the initial color at endpoint v_n, while endpoints v_k and v_n maintain adjacency, resulting in the coloring conflict of both endpoints v_k and v_n being replaced with the initial color.

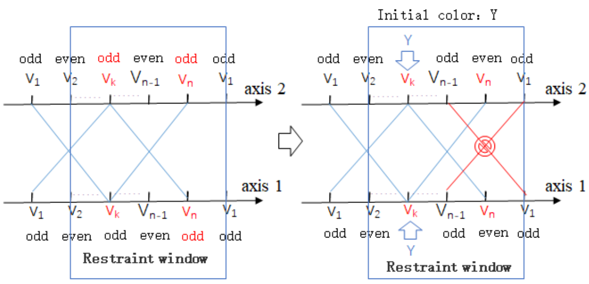

Figure 19: Combination case of four-colorable plane graph number axis value relationship

The above situation only needs to change the replacement of the initial color at endpoint v_k to the insertion of the initial color. As shown in Figure 19, the number axis value relationship of four-colorable plane graph under the combination situation, after inserting the initial color Y, the odd and even colors of the subsequent endpoints are displaced as a whole, which is conducive to breaking the conflict of the odd and even colors of the starting endpoint and terminating endpoint adjacent to the original odd loop. That is, the starting endpoint v_1 and the terminating endpoint v_n on the graph are no longer in (odd, odd) conflict, and the above plane graph coloring evolves into the coloring proof under Case 1 above, so under the combination situation, the plane graph is four-colorable. For the convenience of proof explanation, the model schematic diagram takes the situation of (odd, odd) conflict between the starting endpoint and the terminating endpoint. If the above situation is replaced with (even, even) conflict, as long as the

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corresponding value replacement is done, it also satisfies the proof that the plane graph is four-colorable.

As proven above, through the abstraction of adjacent relationships on the numerical axis, the surrounding module 1 constructed through the decomposition and construction of planar graphs is shown to be four-colorable.

3.2. Discussion on Four Colorable Surrounding Modules Based on Planar Graph Decomposition

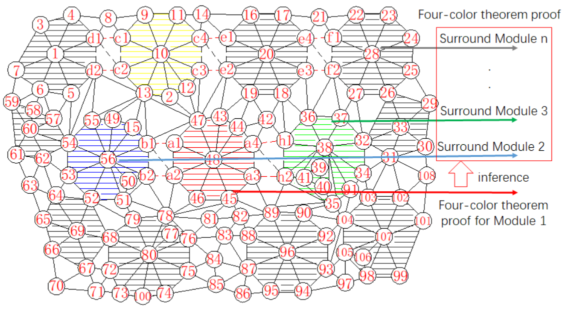

Figure 20: Each surrounding module is four-colorable based on the decomposition of planar graph

Based on the decomposition of planar graphs, the surrounding coloring module 1 is four-colorable; the central endpoint of the surrounding module takes the color red (R), while the circle takes three colors, namely the odd color (blue, L), the even color (green, G), and the initial color (yellow, Y). The colors chosen for the surrounding coloring module 1 are merely numerical substitutes, satisfying a value rotation relationship. Based on the proof of four-colorability for the surrounding coloring module 1, it can be analogously argued that the surrounding coloring modules 2, 3,..., n are also four-colorable under their respective coloring models, as illustrated in Figure 20 showing the individual four-colorability of each surrounding module based on the decomposition of planar graphs.

Therefore, in order to prove that the whole planar graph is four-colorable, it is necessary to demonstrate that the surrounding coloring modules constructed based on the decomposition of planar graphs satisfy the four-colorable constraint relationship of planar graphs, which is demonstrated in the following chapters of this paper.

4. Proof of the Four-Colorability Constraints for Surrounding Modules Based on Planar Graph Decomposition

4.1. Explanatory Notes on Adjacency Relationships of Surrounding Modules Based on Plane Graph Decomposition

4.1.1. The Explanation of Adjacency Relationships for Each Surrounding Module

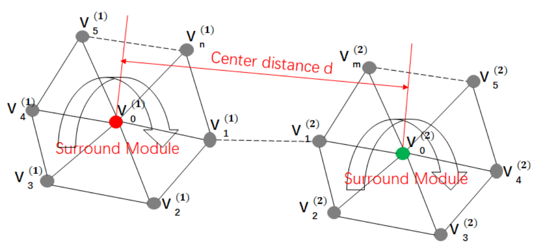

Figure 21: The d-graph model for the central distance of surrounding modules

Based on the planar graph decomposition, we proceed with the graphical construction and modeling for each surrounding module produced. Refer to Figure 21, the d-graph model for the central distance of surrounding modules. In the figure, the central endpoints of two surrounding modules are denoted as $v_0^{(1)}$ and $v_0^{(2)}$, respectively, and their encircling loops are represented by C_n and C_m.The mutual central distance (the distance d between the centers of two surrounding modules, defined as the shortest number of edges between their centers) can be categorized into four scenarios: d=2, d=3, d=4, and d≥5. Ultimately, the adjacency relationships can be reduced to the cases where d=2 and d=3. Below, we will discuss these scenarios in detail.

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4.1.1.1. The case when d=2

Referring to Figure 22, in the case when the central distance d between surrounding modules equals 2, the encircling loops C_n and C_m are adjacent through endpoints $v_1^{(1)}$ and $v_1^{(2)}$. The two endpoints intersect to form a single endpoint. Under this scenario, the two surrounding modules establish an adjacency relationship through their shared endpoint.

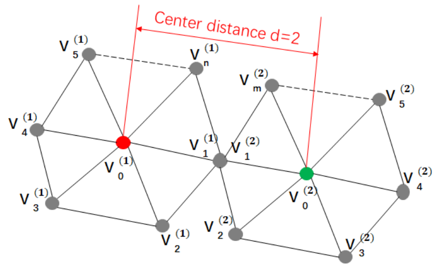

Figure 22: The case of d=2 for the central distance between surrounding modules

4.1.1.2. The case when d=3

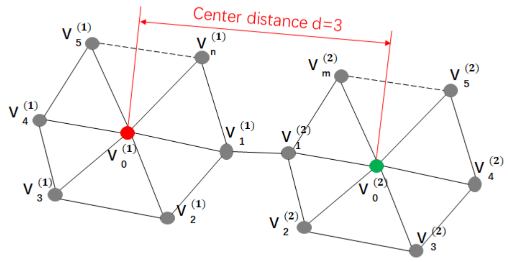

Figure 23: The case of d=3 for the central distance between surrounding modules

As illustrated in Figure 23, in the case when the central distance d between surrounding modules equals 3, the encircling loops C_n and C_m are adjacent through the edge $(v_1^{(1)}, v_1^{(2)})$ connecting endpoints $v_1^{(1)}$ and $v_1^{(2)}$. In this scenario, the two surrounding modules establish an adjacency relationship through this connecting edge.

4.1.1.3. The case when d=4

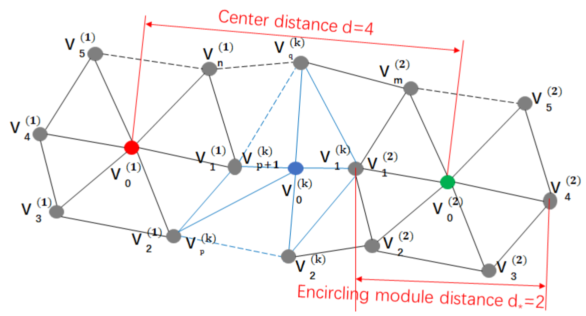

Figure 24: The case of d=4 for the central distance between surrounding modules

As depicted in Figure 24, in the scenario when the central distance d between surrounding modules equals 4, the encircling loops C_n and C_m are adjacent through the edges connecting the endpoints. Specifically, these edges are $(v_1^{(1)}(v_{p+1}^{(k)}), v_0^{(k)})$ and $(v_0^{(k)}, v_1^{(2)}(v_1^{(k)}))$. These edges link the endpoints $v_1^{(1)}(v_{p+1}^{(k)}), v_0^{(k)}, v_1^{(2)}(v_1^{(k)})$, establishing the adjacency relationship between the two encircling modules.In Figure 24, the adjacent edges (emanating from the endpoint $v_0^{(k)}$) namely $(v_0^{(k)}, v_1^{(k)})$ and $(v_0^{(k)}, v_{p+1}^{(k)})$ satisfy the spatial distance requirement for constructing the loop around the module (the inter-vertex distance d_*=2 through the center of the module). Based on the shared common edges between adjacent modules, for instance, the module around $v_0^{(1)}$ shares a common edge with the module around $v_0^{(k)}$, specifically the edges $(v_1^{(1)}, v_2^{(1)})$ and $(v_{p+1}^{(k)}, v_p^{(k)})$. These, however, cannot be constructed by the aforementioned planar graph decomposition rules.

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Based on the decomposition graph of each surrounding module in a planar graph, a connected graph of surrounding modules with common adjacent edges can be constructed when each surrounding module is locally adjacent to d = 4. As in that surround module communication proces of Fig. 25, The surround module around $v_0^{(1)}$ shares a common edge with the surround module around $v_0^{(k)}$ $(v_1^{(1)}, v_2^{(1)})$/ $(v_{p+1}^{(k)}, v_p^{(k)})$, viewed as a locally connected surrounding module, whose center and endpoints are colored the same R = red in Figure 25, This combined surrounding module is adjacent to the surrounding module around $v_0^{(2)}$ by the end point $v_1^{(k)}(v_1^{(2)})$, as in the d = 2 case above (combined center distance d_1=2).

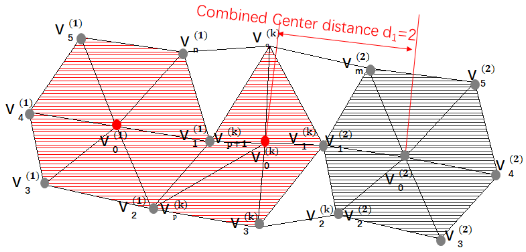

Figure 25: Wraparound module connectivity configuration processing

Based on the combinatorially connected surrounding modules $v_0^{(1)}$, $v_0^{(k)}$, After connection, $v_1^{(1)}/v_{p+1}^{(k)}$ endpoints can be isolated by connecting edges $(v_n^{(1)}, v_q^{(k)})$. That is, the endpoint $v_1^{(1)}/v_{p+1}^{(k)}$ can be regarded as the internal endpoint of the combined connected surrounding module, so its construction area can be filled and covered. As shown in Fig. 26, in the internal filling structure of the combined surrounding module, the triangular regions of $v_n^{(1)}, v_1^{(1)}/v_{p+1}^{(k)}$ and $v_q^{(k)}$ "can be used as the filling and covering structure.

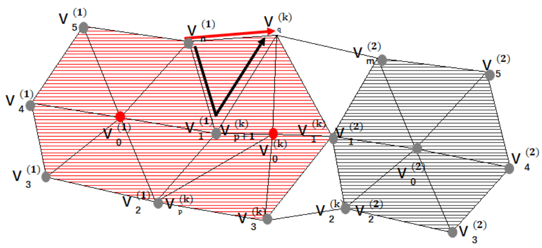

Figure 26: Internal filling construction of combined surrounding modules

4.1.1.4. The case when d≥5

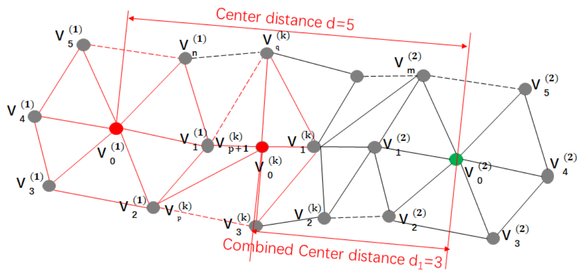

Figure 27: The case of d=5 for the central distance between surrounding modules

Based on the aforementioned adjacency and local connectivity growth construction of surrounding modules, the scenario for surrounding modules with a center distance d=5 is illustrated in Figure 27. The construction of the figure is analogous to the case when d=3 (combined center distance d_1=3), with the combined surrounding modules being adjacent through the edge $(v_1^{(k)}, v_1^{(2)})$. By extension, the construction for the scenario when d=6 is similar to the case when d=4, which in turn is similar to the case when d=2. Therefore, the adjacency relationship of the combined surrounding module center distances can be calculated as follows:

$d_1 = 2 + d/(n \times d_*) = 2 + d \bmod 2$

d: the center-to-center distance of the surrounding modules;

d_*: the construction distance for the surrounding module graph, when d_*=2;

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n: the number of constructible surrounding modules, which is a positive integer;

d_1: the center-to-center distance of the combined surrounding modules;

mod: the modulo operation, which generates an equivalent space;

The modulo operation of the calculated result of d_1 yields a modular space of {2, 3}, thus allowing the connectivity growth patterns of local configurations in each surrounding module to be transformed into cases of adjacency when d=2 and d=3. The adjacency relationships between the surrounding modules are endpoint adjacency and edge adjacency, respectively.

4.1.2. Explanation of the triangular adjacency relationships between each circumferential module

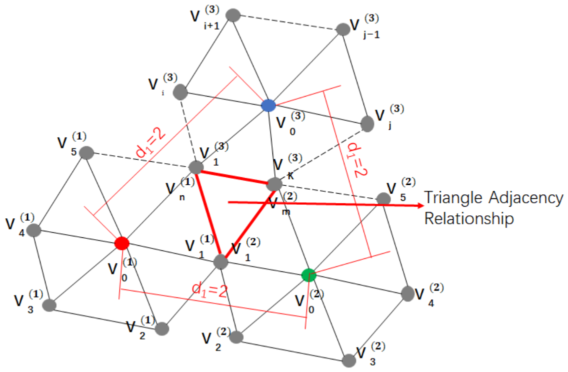

Figure 28: Triangular adjacency relationships for combined surrounding modules with a center-to-center distance d_1=2

Based on the adjacency relationships established by the surrounding module construction, there are two types of adjacency between the modules: endpoint adjacency (d_1=2) and edge adjacency (d_1=3). Due to the characteristics of a maximal planar graph, all endpoints within the plane are in a triangular adjacency, which allows for the construction of triangular adjacency relationships among the surrounding modules. This is illustrated in Figure 28, which shows the triangular adjacency relationships for combined surrounding modules with a center-to-center distance d_1=2, and in Figure 29, which demonstrates the triangular adjacency relationships for combined surrounding modules with a center-to-center distance d_1=3. Below, we will explain these relationships.

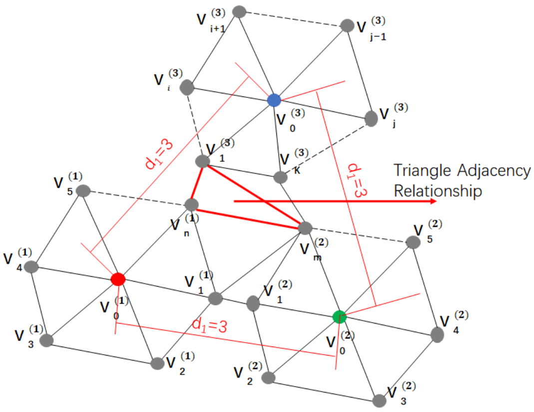

Figure 29: Triangular adjacency relationships for combined surrounding modules with a center-to-center distance d_1=3

As shown in Figure 28, under the condition when the center-to-center distance of the combined surrounding modules is d_1=2, the three surrounding modules adjacent to the central endpoints $v_0^{(1)}, v_0^{(2)}$ and $v_0^{(3)}$ form triangular adjacency relationships through the endpoint pairs $v_1^{(1)}$ and $v_1^{(2)}$, $v_n^{(1)}$ and $v_1^{(3)}$, $v_m^{(2)}$ and $v_k^{(3)}$.As illustrated in Figure 29, under the condition when the center-to-center distance of the combined surrounding modules is d_1=3, the three surrounding modules adjacent to the central endpoints $v_0^{(1)}$, $v_0^{(2)}$, and $v_0^{(3)}$ form triangular adjacency relationships through the adjacent edges $(v_n^{(1)}, v_m^{(2)})$, $(v_n^{(1)}, v_1^{(3)})$, and $(v_m^{(2)}, v_1^{(3)})$.The adjacency conditions for the other combinations of surrounding modules are various combinations of the aforementioned cases with d_1=2 and d_1=3.

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Triangular adjacency relationships for each of the surrounding modules can be constructed based on these combinations.

4.1.3. The triangular adjacency situations of each surrounding module are converted into relationships with d_1=3

Based on the various combinations of triangular adjacency in the surrounding modules mentioned in the preceding text, this includes the various combinations of d_1=2 and d_1=3. To facilitate the proof of the four-coloring of adjacent planes in subsequent sections, this paper follows a unified construction method based on the separation rules of the graphical representation. Specifically, for the construction of d_1=3 triangular adjacency relationships, it is based on the combinations of odd and even loops in the triangular adjacency of the surrounding modules, which can be divided into four situations as shown in the figure.

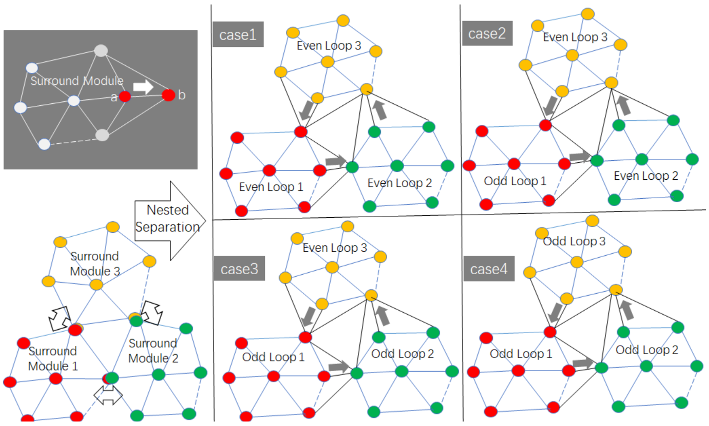

Figure 30: Construction of the separation rule for d_1=2 triangular adjacency relations transformed into the case of d_1=3

The aforementioned separation rules are constructed based on the assumption that merging separated endpoints does not introduce additional adjacency constraints. Nested graph separation is employed for this purpose, as depicted in the left gray box area and the right gray arrow section of Figure 30. Given the relationship between graphically separated endpoints, pairs of separated endpoints, such as a and b, can be colored the same on the loops of the surrounding modules.

4.1.3.1. Graphical Construction Transformation of d_1=2 Triangular Adjacency Relationships

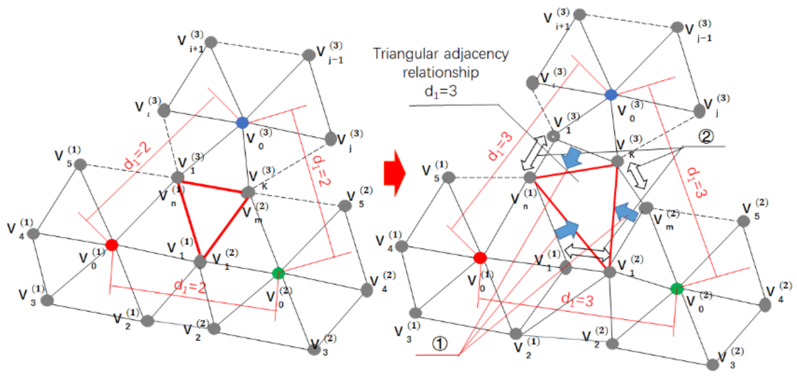

Figure 31: Graphical construction transformation of d_1=2 triangular adjacency relationships

As illustrated in Figure 28, the triangular adjacency relationships with a center-to-center distance d_1=2 for each combination of surrounding modules can, based on the separation rule depicted in Figure 30, be converted into triangular adjacencies with d_1=3, as shown in the graphical construction transformation of d_1=2 triangular adjacency relationships in Figure 31.In Figure 31, on the left side where the red triangles are adjacent, the center-to-center distance between each surrounding module is d_1=2. On the right side, the transformation is achieved through:

①Separation of Shared Adjacent Edges (where the blue unidirectional arrows are): The shared edge $(v_1^{(3)}, v_k^{(3)})$ is separated into the pair of edges $(v_1^{(3)}, v_k^{(3)})$and $(v_n^{(1)}, v_k^{(3)})$. The shared edge $(v_1^{(1)}, v_n^{(1)})$ is separated into the pair of edges $(v_1^{(1)}, v_n^{(1)})$ and $(v_1^{(2)}, v_n^{(1)})$. The shared edge $(v_1^{(2)}, v_m^{(2)})$ is separated into

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the pair of edges $(v_1^{(2)}, v_m^{(2)})$ and $(v_1^{(2)}, v_k^{(3)})$.

②Separation of Shared Adjacent Endpoints (where the black bidirectional arrows are): The shared endpoint $v_1^{(3)}/v_n^{(1)}$ is separated into $v_1^{(3)}$ and $v_n^{(1)}$. The shared endpoint $v_1^{(1)}/v_1^{(2)}$ is separated into $v_1^{(1)}$ and $v_1^{(2)}$. The shared endpoint $v_m^{(2)}/v_k^{(3)}$ is separated into $v_m^{(2)}$ and $v_k^{(3)}$.

Through the above graphical construction, the transformations are uniformly converted into triangular adjacency relationships between the surrounding modules with a center-to-center distance of $d_l=3$.

4.1.3.2. The graphical construction transformation for the combination scenarios of triangular adjacency relationships with $d_l=2$ and $d_l=3$

For other adjacency combinations of the surrounding modules, i.e., the combinations of the surrounding module center distances with $d_l=2$ and $d_l=3$, see the construction of adjacency combinations for the surrounding modules as depicted in Figure 32. In the figure, the surrounding module 1, surrounding module 2, and surrounding module 3 form adjacencies through red triangles.

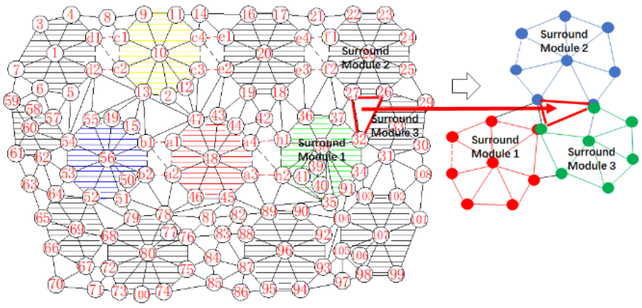

Figure 32: The construction of adjacency combination scenarios for the surrounding modules

In Figure 33, where the adjacency combinations are transformed into triangular adjacency relationships with $d_l=3$, at the location of the red triangular adjacency on the left side: The center-to-center distance between surrounding module 2 and surrounding module 3 is $d_{l-1}=2$;The center-to-center distance between surrounding module 1 and surrounding module 3 is $d_{l-2}=2$;The center-to-center distance between surrounding module 1 and surrounding module 2 is $d_{l-3}=3$; Through the separation rules illustrated on the right side of Figure 33, in accordance with Figure 30, the adjacency relationships can be uniformly transformed into triangular adjacency relationships between the surrounding modules with a center-to-center distance of $d_l=3$.

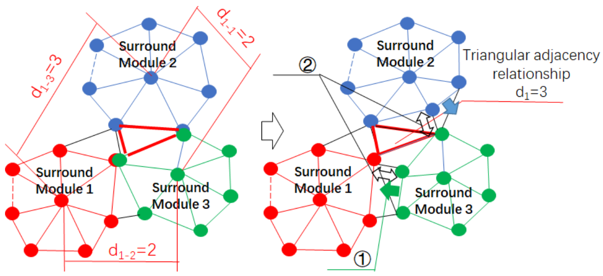

Figure 33: The transformation of adjacency combination scenarios into triangular adjacency relationships with $d_l=3$

4.1.4. Titles The triangular adjacency relationship constructions of each surrounding module

Based on the aforementioned explanation, each surrounding module can be unifiedly converted into a triangular adjacency relationship with a central distance $d_l=3$, generating corresponding adjacent endpoints and adjacent edges. As illustrated in the planar diagram of the surrounding module's peripheral adjacency relations (see Figure 34), taking the surrounding module 1 as an example, the mutual adjacency relations between surrounding module 1 and surrounding modules 2 and 3 generate the starting adjacent endpoint P_1 and the ending adjacent endpoints P_j, P_k, corresponding to the starting adjacent endpoint v_1 and the ending adjacent endpoints v_j, v_k in the diagram. Surrounding module 1 forms triangular adjacency relations with surrounding modules 2 and 3 through the endpoints v_1, and two-to-two adjacency relations are formed with surrounding modules 2 and 3 through adjacent edges $(v_1, ..., v_k)$ and $(v_1, ..., v_j)$, respectively. This process constructs the positional mapping relationships between each surrounding module's adjacency.

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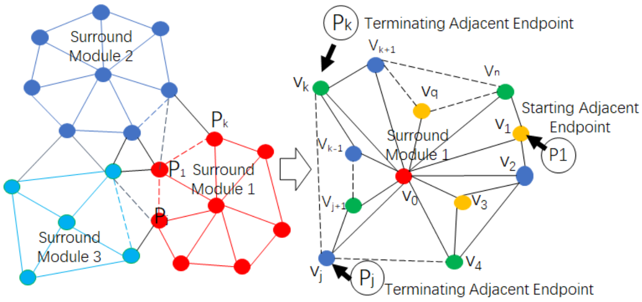

Figure 34: Planar diagram of external adjacency relationships of surrounding modules

4.2. Rotation Relations of Four Colorable Equivalent Classes for Each Surrounding Module Based on Planar Graph Partition

An equivalence relation is a binary relation defined in a set, which must satisfy the following three properties: reflexivity (any element is equivalent to itself), symmetry (if a is equivalent to b, then b is equivalent to a), and transitivity (if a is equivalent to b, and b is equivalent to c, then a is equivalent to c). Based on the coloration relationship argumentation for the circumferential coloring modules discussed above, the color value set for the terminal points of circumferential module 1 can be A1={Central color (Red R), Odd color (Blue L), Even color (Green G), Initial color (Yellow Y)}, with the terminal points of the same color satisfying the equivalence relation. The color value selection for each circumferential module satisfies a rotation relationship; that is, the color value set for the terminal points of circumferential module 2 can be A2={Central color (Red L), Odd color (Blue G), Even color (Green Y), Initial color (Yellow R)}, and the terminal points of the same color also satisfy the equivalence relation. This principle can be extended as required. According to the coloration relationships of terminal points for each loop based on the planar graph decomposition, as long as the same color is taken, the equivalence relation is satisfied. Therefore, the coloration relationships of terminal points for each loop based on the planar graph decomposition can be partitioned into equivalence classes, which means that if one element from the set is chosen, all elements that are equivalent to this chosen element form an equivalence class.

The equivalence relationship of circle coloring based on planar graph partitioning can create an equivalence relation mapping, specifically as follows:

$a \in A, \varphi(a) \in A, b \in B, \varphi(b) \in B, A=B, \phi(a)=b;$

A={central color (red, R), odd-numbered color (blue, L), even-numbered color (green, G), initial color (yellow, Y)};

a is an element within the set A;

B={∇,◇,○,☆};

b is an element within the set B;

φ, ψ, ϕ are mapping functions on the set;

Through the aforementioned transformations, map the value of element a in set A to the equivalence class value of element b in set B, facilitating a generalized proof in the following text.

4.3. Rotation Relations of Four Colorable Equivalent Classes for Each Surrounding Module Based on Planar Graph Partition

For ease of explanation and without loss of generality, a diagram of the adjacency relationships for the elements of the annular configuration, as shown in Figure 35, is created. In this diagram, nodes v_1, v_2, v_3, v_4, v_5, v_6, v_7, ..., v_j, ..., v_k, ..., v_{n-1}, v_n are arranged around the central node v_0. Specifically, node v_1 is adjacent to node v_3, node v_4 is adjacent to node v_7, and node v_j is adjacent to node v_{n-1}. Due to the adjacency relationships within the annulus, the nodes that can be adjacent outside the annulus are v_1, v_3, v_4, v_7, ..., v_j, v_{n-1}, v_n. The external adjacency relationships shown in the diagram are a result of the spatial adjacency constraints within the annulus itself. Given that it has been proved that each surrounding module can be colored with four colors and the number of colors for the annular node endpoints is three, this section attempts to re-color the constructed surrounding modules following certain rules and seeks to identify any patterns.

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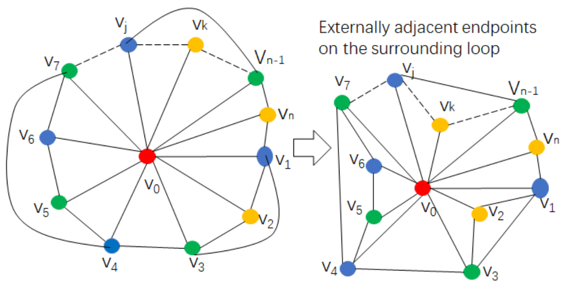

Figure 35: Composition of adjacency relationships between terminal points on a surrounding ring

Based on the adjacency relationships for the endpoints of the annular configuration depicted in Figure 35, the coloring relations for the endpoints within the annulus are constructed as shown in Figure 36, Coloring Relationships for the Endpoints of the Planar Annulus 1. The endpoints outside the annulus are sequentially colored with odd-numbered colors (blue L) and even-numbered colors (green G). Only when the number of endpoints within the annulus is odd, do we need to apply an initial color (yellow Y). In this case, as shown in Figure 36, endpoints v_1, v_3, v_4, v_7, ..., v_j, v_{n-1}, v_n are all odd in number, so v_n requires to be initially colored with yellow Y. If the number of endpoints within the annulus is even, then only odd-numbered colors (blue L) and even-numbered colors (green G) are sequentially applied for coloring, as illustrated in Figure 37, Coloring Relationships for the Endpoints of the Planar Annulus 2. The proof for maintaining the coloring relationships for the endpoints outside the annulus after applying this rule is as follows: The endpoints outside the annulus and the isolated endpoints within the annulus form a circle. If the number of endpoints on this circle is even, the coloring relationship within the annulus is that odd-numbered colors (blue L) are followed by even-numbered colors (green G), which does not disrupt the coloring relationships for the endpoints outside the annulus, as seen in Figures 36 and 37, Circle 2. If the number of endpoints on this circle is odd, then only within the circle, the coloring relationship of odd-numbered colors (blue L) followed by even-numbered colors (green G) is maintained, and an initial color (yellow Y) is inserted without disrupting the coloring relationships for the endpoints outside the annulus, as seen in Figures 36 and 37, Circles 1, 3. If there are still isolated endpoints within the next lower layer of the circle, similar graphical constructions can be used to prove this. Since the coloring treatment for the endpoints outside the annulus satisfies the color permutation relationship for the surrounding modules, and during the coloring process, the endpoints within the annulus only used up to three colors, with the central endpoint colored in one color, the entire surrounding module used up to a maximum of four colors. This meets the previously proven four-colorability for the surrounding modules.

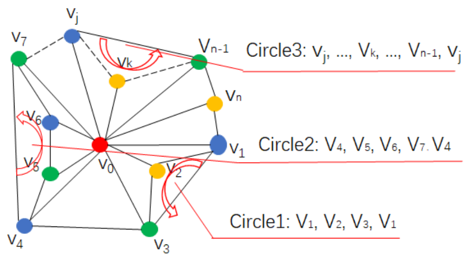

Figure 36: Coloring relationships for the endpoints of the planar annulus 1

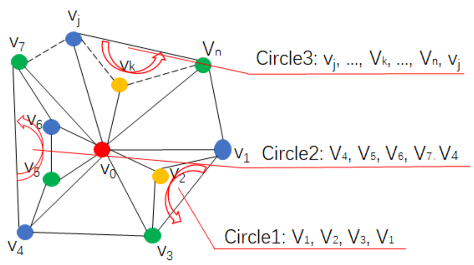

Figure 37: Coloring relationships for the endpoints of the planar annulus 2

Based on the adjacency relationships for colorable and value-switching equivalence for the outer adjacent endpoints of the annular modules as depicted in Figures 36 and 37, with a coloring value domain being the set B (B={▽, ◇, ○, ☆}), when the number of outer annular endpoints is odd, only the terminal endpoint is colored with ▽, while the rest of the endpoints are colored with ◇ and ○ respectively.

Taking into account the adjacency relationships for the endpoints within the annulus, as constructed based on the diagram in Figure 35, we simplify the endpoint coloring by fixing the starting adjacent endpoint v_1 and the terminal adjacent endpoints v_j and v_k as invariant endpoints. As shown in Figure 38, the simplified coloring equivalence partition for the endpoints within the annulus, particularly for the odd annulus, the equivalence classes for coloring values of the outer adjacent endpoints can be divided into:

$$\triangledown=\{v_1\};\Diamond=\{v_2,,,v_k,,,v_{j+1},,,\};\circ=\{v_4,,,v_j,,,v_n\};$$

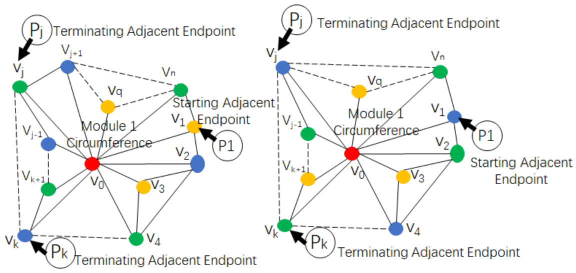

Figure 38: Simplification of coloring equivalence partition for endpoints within a ring in a plane graph

In the figure, the right-hand annulus (an even annulus) has its outer colorable endpoint coloring value equivalence classes divided as follows:

$$\Diamond=\{v_1,v_4,,,v_j,,,\};\circ=\{v_2,,,v_k,,,v_n\};$$

For each annular module, if the number of outer adjacent endpoints is not odd (i.e., it is even), we add the initial color to the coloring on the even ring, ensuring there is exactly one endpoint colored with this initial color. This approach identifies a pattern for coloring the outer endpoints of each annular module. By further unifying the coloring of each annular module into a triangle adjacency coloring with a central distance $d_1=3$, we can systematically simplify the construction of possible configurations, thereby proving that each annular module in a plane graph can be colored in such a way that it is adjacent to at most four other modules.

4.4. Proof of Four-Colorability for Triangular Adjacent Planar Surfaces of Various Surrounding Modules Based on Rule Simplification

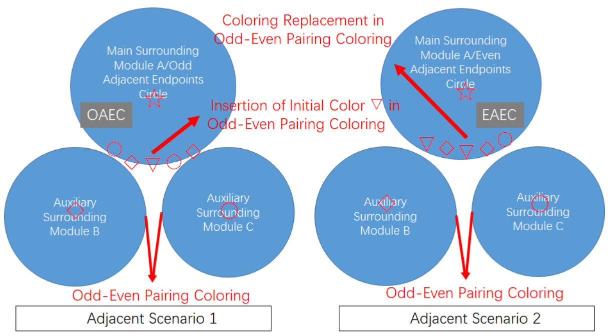

Figure 39: Modeling of adjacency relationships for various surrounding modules

Based on the aforementioned rules for simplifying construction, the adjacency relationships between the modules in the plane graph can be modeled in a graphical format, as shown in the modeling of the adjacency relationships in Figure 39. The adjacency relationships are divided into case 1 and case 2. Case 1 involves an odd number of adjacent vertices (referred to as odd numbered adjacent end-point circles, OAEC) surrounding module A (designated as the primary surrounding module), which can be adjacent to auxiliary surrounding modules B and C. Module A inserts the initial color ▽ into the coloring at the point where the odd-even pairing coloring occurs, satisfying the coloring diagram of a circle with an odd number of adjacent points that can be adjacent ($P=2n+1$, where P is the number of external adjacent vertices, and n is the number of paired coloring of external vertices). Modules B and C are each assigned one of the two colors at the adjacent points in accordance with the odd-even pairing. Case 2 involves an even number of adjacent vertices (also referred to as even numbered adjacent end-point circles, EAEC) surrounding module A, which can be adjacent to auxiliary surrounding modules B and C. Module A replaces the circle symbol with ▽ in the odd-even pairing coloring at the adjacency point, satisfying the coloring diagram of an even number of adjacent points that can be adjacent ($P=2n$, where P is the

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number of external adjacent vertices, and n is the number of paired coloring of external vertices). Modules B and C are each assigned one of the two colors at the adjacent points in accordance with the odd-even pairing.

Based on the adjacency relationship modeling above, a unified and standardized pattern for coloring the adjacent endpoints of each surrounding module has been established. Below, we apply this rule to prove the four-colorability of the adjacency situations of the surrounding modules in the constructed configuration in a plane.

4.4.1. The three-dimensional axis relationship under triangular adjacency with a central distance of $d_1=3$ is adopted

Based on the triangular adjacency situations of each surrounding module as discussed in section 4.1.3, these can be converted into relationships where $d_1=3$, allowing for the construction of an adjacent three-dimensional axis relationship, as illustrated in Figure 40. The three-dimensional axis relationships for the adjacency of each surrounding module are adopted in this construction. In Figure 40, the coloring of the adjacent endpoints of surrounding module 1 takes values on axis 1, the coloring of the adjacent endpoints of surrounding module 2 takes values on axis 2, and the coloring of the adjacent endpoints of surrounding module 3 takes values on axis 3. This converts the issue of adjacent coloring for each surrounding module into a problem of taking values on each axis. Based on the definition of equivalent classes for coloring values of connectable endpoints outside the circle as stated previously, the value space for axis 1 is three colors from the coloring value domain B (where B={▽,◇,○,☆}), set as {▽=Initial color,◇=Odd color,○=Even color}. The coloring value domains for axis 2 and 3 can be the other three colors from B (where B={▽,◇,○,☆}) respectively.

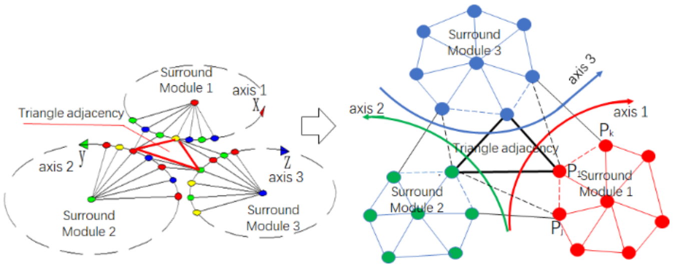

Figure 40: Illustrates the adoption of three-dimensional axis relationships for the adjacency of each surrounding module

4.4.2. The combinations of coloring endpoint sets for adjacency cases 1 and 2

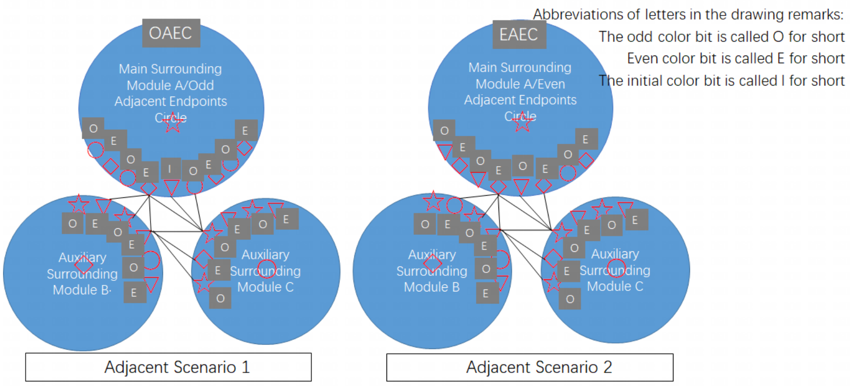

Figure 41: The combinations of the coloring endpoints for each adjacent surrounding module

Based on the adjacency situation 1 pattern in Figure 39 and the three-dimensional coordinate relationship of adjacency among surrounding modules adopted in Figure 40, we construct combinations of coloring for the adjacent endpoints of each surrounding module, as shown in the array combinations of colored endpoints in Figure 41. In the OAEC's endpoints, the initial color {▽} is inserted into the coloring in the odd color region {○} and even color region {◇} in the triangular adjacency area, while all other adjacent endpoints of surrounding modules are alternately colored with odd and even colors. In the EAEC's endpoints, the even color {▽} is replaced in the odd color region {○} and even color region {◇} in the triangular adjacency area, while all other adjacent endpoints of surrounding modules are alternately colored with odd and even colors. Through the coloring rules applied to the adjacency of each pair of surrounding modules, the maximum number of coloring categories is three, including the initial color, odd color, and even color, or odd/even replacement color, odd color, and even color. Through the adjacency relationships, we generate an equivalent relationship K for adjacency positions. Based on the proof of four-colorability for each surrounding module, the maximum number of colors for a surrounding module when it is not adjacent is three, which produces an equivalent relationship S for endpoint coloring values. The coloring of adjacent endpoints of surrounding modules becomes a sequence of these relationships, represented as (K, S) sequences, for example, in adjacency situation 1 where the main surrounding module A is adjacent to the auxiliary surrounding module C, the (K, S) sequence for the endpoints adjacent to module A is (even, ◇), (Initial, ▽), (odd, ○), and the same construction applies to the coloring of other endpoints. Due to the normalization of the coloring rules for adjacency positions and the constraint on the number of endpoint colors, the number of endpoint values for adjacency positions in the triangular adjacency area for each surrounding module A, B, and C is equivalent: A=B=C={1, 2, 3}.

Due to the rotational combination relationship satisfied by surrounding modules A, B, and C, there are 10 such combination situations. The coloring combinations for the endpoints of each surrounding module under adjacency case 1 are as follows:

C1={1,1,1};C2={1,1,2};C3={1,1,3};C4={1,2,2};C5={1,2,3};

C6={1,3,3};C7={2,2,2};C8={2,2,3};C9={2,3,3};C10={3,3,3};

Similarly, the coloring combinations for the endpoints of each surrounding module under adjacency case 2 are as follows:

C11={1,1,1};C12={1,1,2};C13={1,1,3};C14={1,2,2};C15={1,2,3};

C16={1,3,3};C17={2,2,2};C18={2,2,3};C19={2,3,3};C20={3,3,3};

4.4.3. The proof of four-colorability for adjacency case 1 in the plane

4.4.3.1. Four-color theorem proof for the C1={1,1,1} case in a planar graph

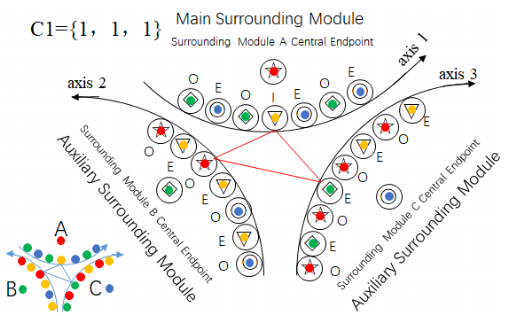

Figure 42: Coloring relationship between adjacent surrounding module circles for C1={1, 1, 1}

In order to prove that the planar graph can be four-colored under each adjacency combination of surrounding modules in case 1 on the left side of Figure 41, we adopt the three-axis diagram model as shown in Figure 40. This is demonstrated in Figure 42, where the coloring relationship of surrounding modules adjacent to the circle of C1={1,1,1} is depicted.In the figure, the structure of the surrounding module A is constructed as the main surrounding module, and the adjacent edges with corresponding surrounding modules are constructed as axis 1.The structure of the surrounding module B is constructed as the auxiliary surrounding module, and the adjacent edges with corresponding surrounding modules are constructed as axis 2.Likewise, the structure of the surrounding module C is also constructed as an auxiliary surrounding module, and the adjacent edges with corresponding surrounding modules are constructed as axis 3.This is all under the condition that the adjacency center distance for each surrounding module is d_1=3.

The endpoint coloring value domain is B={△=red(R), ◇=blue(L), ○=green(G), ☆=yellow(Y)}; The letters in the figure are abbreviated as O for odd color bit, E for even color bit, and I for initial color bit. For the convenience of writing in the figure, the subsequent figures are uniformly abbreviated according to this.

In the primary surrounding module A's axis 1, we insert a symbol {▽} in the middle of the successive pairing of odd {◇} and even {○} colors. This represents the coloring structure in the circumferential

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ring outside, satisfying P=2n+1, for the odd endpoints that can be adjacent. Here, the inserted {▽} is considered as the initial color. Based on the coloring structure mentioned above, it has only one endpoint. The central endpoint of the main surrounding module is colored as {☆}.The coloring sequence is denoted as A (initial color, ▽; odd, ◇; even, ○; central color, ☆), complying with the four-coloring rule.

The auxiliary surrounding modules B and C perform corresponding color pairing at the endpoints in the adjacent areas. The color pairing should adopt the odd-even pairing rule, which means odd and even alternation successively. Such as axis 2: B (odd, ☆; even, ▽). In this case, the odd color can be replaced correspondingly, turning into B (odd, ○; even, ▽), without changing the odd-even alternation. Axis 3: C (odd, ☆; even, ◇). here the even color can be replaced correspondingly, turning into C (odd, ☆; even, ▽), also without changing the odd-even alternation. Based on the above axis coloring value construction, the requirement of needing two colors to be paired at the endpoints in the adjacent areas is satisfied. This complies with the structure in adjacency situation 1 as shown in Figure 41.

The auxiliary surrounding module B and the main surrounding module A in Figure 42 have axis 1 in the negative axis direction and axis 2 in the positive axis direction. The coloring of axis 2 for B is (odd, ☆; even, ▽), which is different from the adjacent axis 1 coloring for A (odd, ◇; even, ○). This satisfies the color relationship for adjacent endpoints.

The auxiliary surrounding module C and the main surrounding module A in Figure 42 have axis 1 in the positive axis direction and axis 3 also in the positive axis direction. The coloring of axis 3 for C is (odd, ☆; even, ▽), which is different from the adjacent axis 1 coloring for A (odd, ◇; even, ○). This also satisfies the color relationship for adjacent endpoints.

The auxiliary surrounding module B and the auxiliary surrounding module C in Figure 42 have axis 2 in the negative axis direction and axis 3 in the negative axis direction. The coloring of axis 2 for B is (odd, ○; even, ▽), and the coloring of axis 3 for C is (odd, ☆; even, ◇). These colorings are different from each other, satisfying the color relationship for adjacent endpoints.

The central endpoint of the auxiliary surrounding module B is colored as B (central color, ◇), complying with the four-color rule. The central endpoint of the auxiliary surrounding module C is colored as C (central color, ○), also complying with the four-color rule.

The above endpoint colorings for adjacent areas satisfy the requirements for the structure in adjacency situation 1 as shown in Figure 41. Based on the above proof, the adjacency conditions for each surrounding module under C1={1,1,1} satisfy the four-colorability of the planar graph.

4.4.3.2. Four-color theorem proof for the C10={3,3,3} case in a planar graph

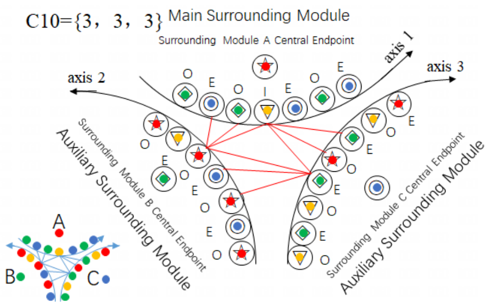

Figure 43: Coloring relationship between adjacent surrounding module circles for C10={3, 3, 3}

In Figure 43, the number of endpoints for the adjacent modules A, B and C is C10={3,3,3}, and the composition coloring of the main surrounding module A is the same as that of C1, satisfying four-colorability;

The auxiliary surrounding modules B and C have corresponding colorings at the endpoints in the adjacent areas. The colorings must adopt successive odd and even pairings. For instance, on axis 2, B (odd, ☆; even, ▽) can have even replacements, becoming B (odd, ☆; even, ○), but without changing the succession of odd and even. On axis 3, the front section is C (odd, ▽; even, ◇), the middle section can have odd replacements, becoming C (odd, ☆; even, ◇), but without changing the succession of odd and even. The tail section can have even replacements for the front section, becoming C (odd, ▽; even, ☆), but without changing the succession of odd and even. Based on the above construction of axis

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coloring values, it satisfies the requirement that endpoints at adjacent areas need to be paired with two colors, conforming to case 1 of adjacency in Figure 41.

The auxiliary surrounding module B and the main surrounding module A in Figure 43 have axis 2 in the positive axis direction, colored as B (odd, ☆; even, ▽), and axis 1 adjacent to it is colored as A (odd, ◇; even, ○). The colors are different, satisfying the relationship for adjacent endpoints.

The auxiliary surrounding module C and the main surrounding module A in Figure 43 have axis 3 in the positive axis direction, colored as C (odd, ▽; even, ☆), and axis 1 adjacent to it is colored as A (odd, ◇; even, ○). The colors are different, satisfying the relationship for adjacent endpoints.

The auxiliary surrounding module B and the auxiliary surrounding module C in Figure 43 have axis 2 in the negative axis direction, colored as B (odd, ☆; even, ○), and axis 3 in the negative axis direction, colored as C (odd, ▽; even, ◇). Their colors are different, satisfying the relationship for adjacent endpoints.

The central endpoint of the auxiliary surrounding module B is colored as B (central color, ◇), satisfying the four-colorability. The central endpoint of the auxiliary surrounding module C is colored as C (central color, ○), satisfying the four-colorability.

The above endpoint colorings for adjacent areas meet the requirements for case 1 of adjacency as shown in Figure 41. Based on the above proof, the adjacency conditions for each surrounding module under C10={3,3,3} satisfy the four-colorability of the planar graph.

4.4.3.3. Display of the proof for four-colorability in planar graphs for scenarios C1 through C10

The endpoint coloring combinations for the surrounding modules under adjacency condition 1 in Figure 41 are denoted as C1 to C10. The four-color theorem proof for conditions C1 and C10 have already been presented in the preceding text. Similar proofs can be conducted for the remaining conditions, but will not be elaborated on specifically. Refer to Figure 44 for the coloring relationships between adjacent surrounding modules in C1 to C10; all combinations are listed and the proof is demonstrated through coloring.

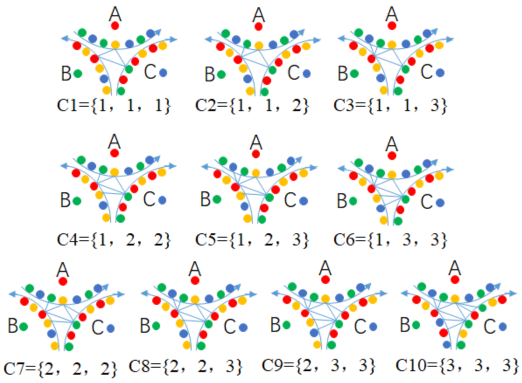

Figure 44: Depicts the coloring relationships between adjacent loops of surrounding modules from C1 to C10

4.4.4. The proof of four-colorability for adjacency case 2 in the plane

4.4.4.1. Four-color theorem proof for the C11={1,1,1} case in a planar graph

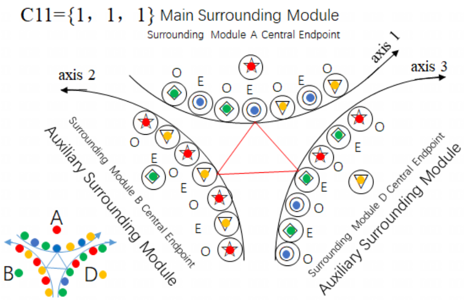

Figure 45: Coloring relationship between adjacent surrounding module circles for C11={1, 1, 1}

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In order to prove the four-colorability of planar graphs under adjacency condition 2 on the right side of Figure 41 for the adjacent endpoint combinations of each surrounding module, refer to the coloring relationship between adjacent surrounding modules in Figure 45 for C11={1,1,1}. In the figure, encircling module A is constructed as the main surrounding module, and axis 1 is constructed along the adjacent sides with the corresponding surrounding modules; surrounding module B is constructed as an auxiliary surrounding module, and axis 2 is constructed along the adjacent sides with the corresponding surrounding modules; surrounding module D is constructed as an auxiliary surrounding module, and axis 3 is constructed along the adjacent sides with the corresponding surrounding modules. The adjacent central distances for each surrounding module are d_1=3.

On axis 1 of the main surrounding module A, {◇} in the successive odd {◇} and even {○} pair coloring is replaced by {▽}, forming a new pair coloring, which satisfies the coloring configuration for the outer ring of the circle when P=2n for even endpoints that can be adjacent. The center endpoint of the main encircling module is colored {☆}, satisfying the four-color theorem.

The auxiliary surrounding modules B and D have corresponding colorings at the endpoints in the adjacent areas. The colorings must adopt successive odd and even pairings. For axis 2, B (odd, ☆; even, ▽); axis 3, D (odd, ☆; even, ◇). The odd colorings in between can be replaced, resulting in D (odd, ○; even, ◇), but without changing the succession of odd and even. Based on the above construction of axis coloring values, it satisfies the requirement that endpoints at adjacent areas need to be paired with two colors, conforming to adjacency case 2 in Figure 41.

The auxiliary surrounding module B and the main surrounding module A in Figure 45 have axis 2 in the positive axis direction with the coloring B (odd, ☆; even, ▽), which differs from the adjacent axis 1 coloring for A (odd, ◇; even, ○), satisfying the relationship between adjacent endpoints.

The auxiliary surrounding module D and the main surrounding module A in Figure 45 have axis 3 in the positive axis direction with the coloring D (odd, ☆; even, ◇), which differs from the adjacent axis 1 coloring for A (odd, ▽; even, ○), satisfying the relationship between adjacent endpoints.

The auxiliary surrounding module B and the auxiliary surrounding module D in Figure 45 have axis 2 in the negative axis direction with the coloring B (odd, ☆; even, ▽), and axis 3 in the negative axis direction with the coloring D (odd, ○; even, ◇). Their colors differ, satisfying the relationship between adjacent endpoints.

The center endpoint of the auxiliary surrounding module B is colored B (central color, ◇), satisfying the four-color theorem. The center endpoint of the auxiliary surrounding module D is colored D (central color, ▽), satisfying the four-color theorem.

The above endpoint colorings for adjacent areas satisfy the requirements for adjacency case 2 as shown in Figure 41. Based on the above proof, the adjacency conditions for each surrounding module under C11={1,1,1} satisfy the four-colorability of the planar graph.

4.4.4.2. Four-color theorem proof for the C20={3,3,3} case in a planar graph

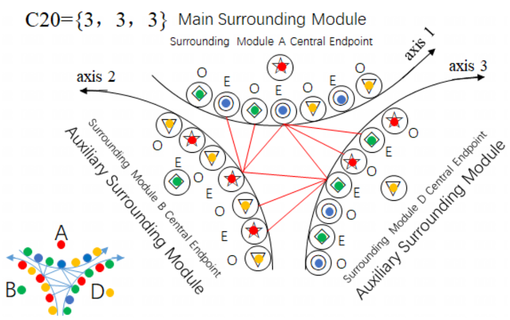

Figure 46: Coloring relationship between adjacent surrounding module circles for C20={3, 3, 3}

Figure 46 demonstrates the scenario where the endpoint values for surrounding modules A, B, and D are in the combination of C20 = {3,3,3}. The configuration and requirements for adjacent endpoint coloring are the same as for C11. Based on this, endpoint coloring is performed for each adjacent surrounding module. The coloring of the main surrounding module is the same as for C11, meeting the requirements of the four-color theorem.

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The auxiliary surrounding module B and the main surrounding module A in Figure 46 have axis 2 in the positive axis direction with the coloring B (odd, ▽; even, ☆), which differs from the adjacent axis 1 coloring for A (odd, ◇; even, ○), satisfying the relationship between the colors of adjacent endpoints.

The auxiliary surrounding module D and the main surrounding module A in Figure 46 have axis 3 in the positive axis direction with the coloring D (odd, ☆; even, ◇), which differs from the adjacent axis 1 coloring for A (odd, ▽; even, ○), satisfying the relationship between the colors of adjacent endpoints.

The auxiliary surrounding module B and the auxiliary surrounding module D in Figure 46 have axis 2 in the negative axis direction with the coloring B (odd, ▽; even, ☆), and axis 3 in the negative axis direction with the coloring D (odd, ○; even, ◇). Their colors differ, satisfying the relationship between the colors of adjacent endpoints.

The center endpoint of the auxiliary surrounding module B is colored B (center color, ◇), meeting the requirements of the four-color theorem.The center endpoint of the auxiliary surrounding module D is colored D (center color, ▽), meeting the requirements of the four-color theorem.

The above-mentioned coloring of the endpoints in the adjacent areas meets the requirements for adjacency case 2 as shown in Figure 41. Based on this proof, the adjacency situation of C20 = {3,3,3} between each surrounding module meets the four-colorability of the planar graph.

4.4.4.3. Display of the proof for four-colorability in planar graphs for scenarios C11 through C20

The endpoint coloring combinations for the surrounding modules under adjacency condition 2 in Figure 41 are represented as C11 to C20. The proof for four-colorability for conditions C11 and C20 have already been provided in the preceding text. Similar proofs can be conducted for the remaining conditions, but will not be elaborated in detail. Refer to Figure 47 for the coloring relationships between adjacent surrounding modules in C11 to C20, which lists all combinations and demonstrates the coloring proof.

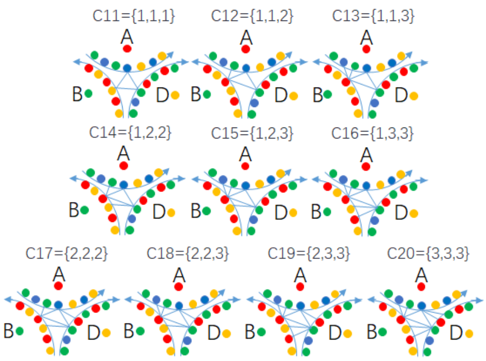

Figure 47: Depicts the coloring relationships between adjacent loops of surrounding modules from C11 to C20

Based on the above proofs for these combinations, it has been proven that under triangular adjacency of surrounding modules, and when the center distance is d_1=3, the planar graph is four-colorable. The following text will use a similar method to prove that the planar graph is also four-colorable when d_1=2.

4.4.5. Proof of four-colorability for the case with a center distance of d_1 = 2

Based on the adjacency conditions depicted in Figure 41, the adjacency conditions under the center distance d_1 = 2 are also categorized into two scenarios, corresponding to Case 1 and Case 2 in Figure 41. These conditions are analyzed in relation to the separation construction illustrated in Figure 30. The endpoint coloring combinations for each surrounding module under adjacency condition 1 are as follows:

For adjacency with 3 separated endpoint pairs: C21 = {3, 3, 3}; For adjacency with 2 separated endpoint pairs: C22 = {3, 3, 3}, C23 = {2, 3, 3}, C24 = {1, 3, 3}; Adjacent separated endpoints as 1 pair (at the interface between the main and auxiliary surrounding modules): C25 = {3, 3, 3}, C26 = {2, 3, 3}, C27 = {2, 2, 3}, C28 = {1, 3, 3}, C29 = {1, 2, 3}, C30 = {1, 1, 3}; Adjacent separated endpoints as 1 pair (at the interface between the two auxiliary surrounding modules): C45 = {3, 3, 3}, C46 = {2, 3, 3}, C47 = {2, 2, 3}, C48 = {1, 3, 3}, C49 = {1, 2, 3}, C50 = {1, 1, 3};

Similarly, the endpoint coloring combinations for each surrounding module under adjacency condition 2 are as follows:

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For adjacency with 3 separated endpoint pairs: C31 = {3, 3, 3}; For adjacency with 2 separated endpoint pairs: C32 = {3, 3, 3}, C33 = {2, 3, 3}, C34 = {1, 3, 3}; For adjacency with 1 separated endpoint pair: C35 = {3, 3, 3}, C36 = {2, 3, 3}, C37 = {2, 2, 3}, C38 = {1, 3, 3}, C39 = {1, 2, 3}, C40 = {1, 1, 3}.

4.4.5.1. Four-color theorem proof for the C21={3,3,3} case in a planar graph

Referring to the coloring relationship between adjacent surrounding modules in Figure 48 for C21 = {3,3,3}, this is a triangular adjacency diagram obtained by the separation construction through Figure 30, where the center distance between surrounding modules is d_1=2. There are three pairs of separated endpoints at the adjacency points, and the axis configuration is the same as in C1. Based on this, endpoint coloring is performed for each adjacent surrounding module. The coloring of the main surrounding module A is the same as in C1, and in axis 1, an initial color {▽} is inserted between the successive odd and even pair colorings, satisfying the coloring configuration for the outer ring of the circle when P=2n+1 for odd endpoints that can be adjacent. The center endpoint of the main surrounding module is colored {☆}, meeting the conditions of the four-color theorem.

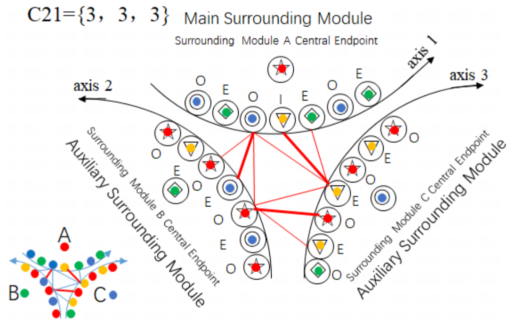

Figure 48: Coloring relationship between adjacent surrounding module circles for C21={3, 3, 3}

The auxiliary surrounding modules B and C undergo corresponding odd and even pair colorings at the endpoints in the adjacent areas, such as B (odd, ☆; even, ▽). The even colorings can be replaced correspondingly, such as B (odd, ☆; even, ○), without changing the odd-even succession characteristics. Based on the above construction of axis coloring values, it satisfies the requirement for paired coloring at the endpoints at adjacent areas, conforming to adjacency case 1 as shown in Figure 41.

The auxiliary surrounding module B and the main surrounding module A in Figure 48 have axis 2 in the positive axis direction with the coloring B (odd, ☆; even, ▽), which differs from the adjacent axis 1 coloring for A (odd, ○; even, ◇), meeting the relationship for adjacent endpoints.

The auxiliary surrounding module C and the main surrounding module A in Figure 48 have axis 3 in the positive axis direction with the coloring C (odd, ☆; even, ▽), which differs from the adjacent axis 1 coloring for A (odd, ○; even, ◇), meeting the relationship for adjacent endpoints.

The auxiliary surrounding module B and the auxiliary surrounding module C in Figure 48 have axis 2 in the negative axis direction with the coloring B (odd, ☆; even, ○), and axis 3 in the negative axis direction with the coloring C (odd, ◇; even, ▽). Their colors differ, meeting the relationship for adjacent endpoints.

Based on the characteristics of the separation endpoints in Figure 30, there are three pairs of separated adjacency endpoints in the triangular adjacency zone marked by the thick red line, which can be colored in the same color, as shown in the figure: The paired terminals of the main surrounding module A and the auxiliary surrounding module B are colored {○}, the paired terminals of the main surrounding module A and the auxiliary surrounding module C are colored {▽}, and the paired terminals of the auxiliary surrounding module B and the auxiliary surrounding module C are colored {☆}. When merged, they form a single color, distinct from the colors at the center of their respective surrounding modules.

The center endpoint of the auxiliary surrounding module B is colored B (central color, ◇), meeting the conditions of the four-color theorem.The center endpoint of the auxiliary surrounding module C is colored C (central color, ○), meeting the conditions of the four-color theorem.

The above endpoint colorings for adjacent areas meet the requirements for adjacency case 1 as shown in Figure 41. Based on the above proof, the adjacency situation of C21 = {3,3,3} between each surrounding module meets the four-colorability of the planar graph.

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4.4.5.2. Four-color theorem proof for the C45={3,3,3} case in a planar graph

Referring to the coloring relationship between adjacent surrounding modules in Figure 49 for C45 = {3,3,3}, this represents a triangular adjacency diagram obtained through the separation construction in Figure 30 for situations where the center distance between encircling modules is d_1=2. There is one pair of separated endpoints at the adjacency points, indicated by the thick red line connecting even-even circles. The axis configuration is the same as in C1. Based on this, endpoint coloring is performed for each adjacent surrounding module. The coloring of the main surrounding module A is the same as for C1, and it meets the conditions of the four-color theorem.

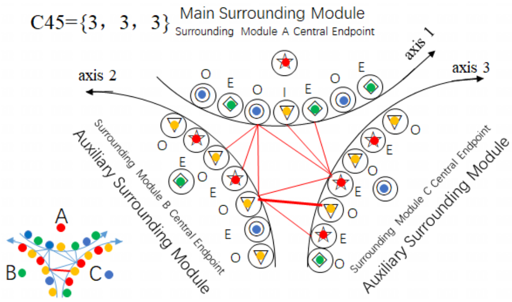

Figure 49: Coloring relationship between adjacent surrounding module circles for C45={3, 3, 3}

The auxiliary surrounding modules B and C undergo corresponding colorings at the endpoints in the adjacent areas. The coloring must adopt the odd-even pair matching technique to satisfy the configuration for adjacency case 1 as shown in Figure 41. The coloring for the adjacency endpoints follows the same rules as for C21, with the paired terminals of the auxiliary surrounding module B and the auxiliary surrounding module C colored {▽}. When merged, they form a single color, distinct from the colors at the center of their respective surrounding modules.

Based on the above proof, the adjacency situation of C45 = {3,3,3} between each surrounding module meets the four-colorability of the planar graph. This demonstrates that the complex adjacency conditions for surrounding modules, when the center-to-center distance is d_1=2, can also be properly addressed with appropriate coloring strategies, ensuring four-colorability.

4.4.5.3. Display of the proof for four-colorability in planar graphs for scenarios C21 through C30 and C45 through C50

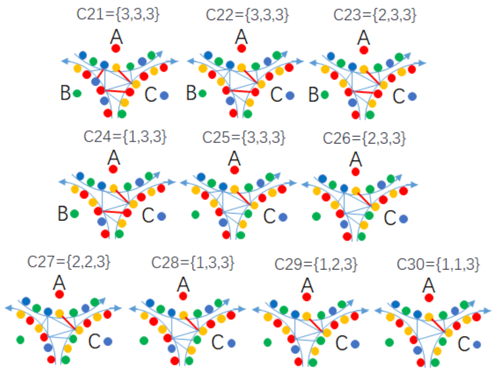

Figure 50: Depicts the coloring relationships between adjacent surrounding modules for the range of combinations from C21 to C30

The endpoint coloring combinations for adjacency condition 1 for each surrounding module are represented as C21 to C30, and C45 to C50. Proof for four-colorability for conditions C21 and C45 have been provided in the preceding text. Similar proofs can be conducted for the remaining conditions but will not be elaborated upon in detail. Refer to Figure 50 for the coloring relationships between adjacent surrounding modules C21 to C30, and Figure 51 for C45 to C50, which list all combinations for a coloring demonstration. In these figures, the pairs of endpoints connected by red adjacency lines denote the separation and merging endpoints.

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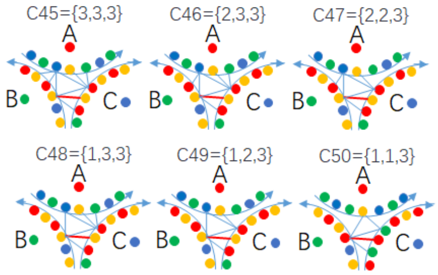

Figure 51: Depicts the relationships of color assignments between surrounding modules that are adjacent, for the combinations ranging from C45 to C50

4.4.5.4. Four-color theorem proof for the C31={3,3,3} case in a planar graph

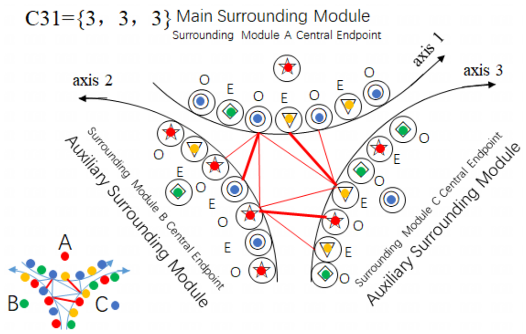

Figure 52: Coloring relationship between adjacent surrounding module circles for C31={3, 3, 3}

Referring to Figure 52 for the coloring relationship of adjacent surrounding modules for C31 = {3,3,3}, this is a triangular adjacency diagram obtained from adjacency condition 2, when the center-to-center distance between encircling modules is d_1=2, through the separation construction in Figure 30.

In the main surrounding module A, for the successive pairing of colors A (odd, ○; even, ◇) in axis 1, the even color is replaced with A (odd, ○; even, ▽), forming a new pairing color. This satisfies the coloring configuration for the outer ring of the circle when P=2n for even endpoints that can be adjacent. The center endpoint of the main surrounding module A is colored A (central color, ☆), meeting the conditions of the four-color theorem.

The auxiliary surrounding modules B and C undergo corresponding colorings at the endpoints in the adjacent areas, which must adopt odd-even pair matching, that is, odd followed by even. For instance, in axis 2: B (odd, ☆; even, ○), the even color can be replaced correspondingly, leading to B (odd, ☆; even, ▽), without changing the odd-even succession. In axis 3: the preceding section is C (odd, ◇; even, ▽), in the middle section of which the odd color can be replaced correspondingly, resulting in C (odd, ☆; even, ▽). The latter section can replace the even color of the preceding section, leading to C (odd, ◇; even, ☆), again without changing the odd-even succession.

Based on the construction of the above axis coloring values, it satisfies the requirement for paired coloring at the endpoints at adjacent areas, conforming to the configuration for adjacency case 2 as shown in Figure 41. The coloring for the adjacency endpoints follows the same rules as for C21, where there are three pairs of separated adjacency endpoints in the triangular adjacency zone marked by the thick red line, which can be colored in the same color, respectively.Based on the above proof, the adjacency situation of C31 = {3,3,3} between each surrounding module meets the four-colorability of the planar graph, demonstrating that the complex adjacency conditions for surrounding modules can be adequately addressed with the appropriate coloring strategies, ensuring four-colorability.

4.4.5.5. Display of the proof for four-colorability in planar graphs for scenarios C31 through C40

The coloring combinations for the endpoints of surrounding modules in adjacency case 2 range from C31 to C40. The proof for four-colorability for combination C31 has already been presented in the foregoing discussion. Analogous proofs can be conducted for the remaining combinations, although these will not be detailed specifically. Refer to Figure 53 for the inter-coloring relations between adjacent surrounding modules from C31 to C40; this figure demonstrates all possible combinations, providing a visual proof of the coloration. In the figure, paired endpoints connected by red adjacency lines represent

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the disjoined and merged endpoints.

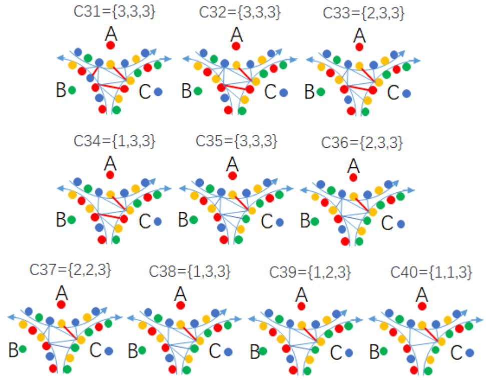

Figure 53: Depicts the relationships of color assignments between surrounding modules that are adjacent, for the combinations ranging from C31 to C40

Based on the handling and proof of coloring for the aforementioned combinations, all combinations satisfy the condition for four-colorability in planar graphs. This validates that all triangular adjacency cases for the surrounding modules constructed in this paper can be four-colorable on a plane.

4.5. Based on the adjacency relations of each surrounding module, prove the four-coloring theorem for every pair of adjacent modules

The previous section has already demonstrated the four-colorability of each triangularly adjacent module junction and the relationship of adjacency among the modules based on the composition and structure described in Section 4.1.4. This section builds upon this foundation to prove the four-colorability of adjacency among each pair of modules. The adjacency scenarios for each pair of modules are divided into cases based on the distance (d_1) between them, specifically for (d_1=3) and (d_1=2) combinations.

Case I deals with the combination of two adjacent modules where one has a single endpoint ((k=1), located at the triangular adjacency junction) and the other has (d_1=3). This case was already proven for four-colorability in the previous section, specifically in the proofs of 4.4.3 and 4.4.4.

Case II focuses on the combination of two adjacent modules where one has a single endpoint ((k=1), not located at the triangular adjacency junction) and the other has (d_1=3).

Case III addresses the combination of two adjacent modules with multiple endpoints ((k≥2)) where one has (d_1=2) and the other has (d_1=3).

4.5.1. The proof of four-colorability for the II case of adjacency between pairs of surrounding modules

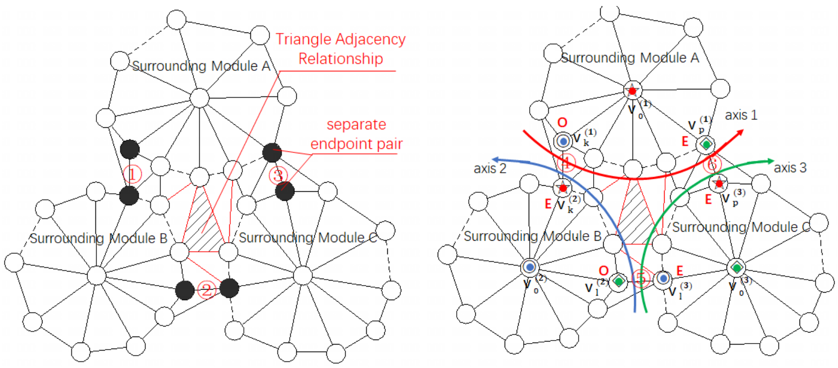

Figure 54: Color combinations for the II case of adjacency between pairs of surrounding modules

The combination with the highest number of adjacent connections for the II case of adjacency between pairs of surrounding modules is where each surrounding module with triangular adjacency has d_1=2 endpoints adjacent to each other, as shown in the coloring combinations of Figure 54. The figure presents a total of three pairs of separated endpoints on the left, representing the maximum number of pairs of separated endpoints in adjacent conditions. Based on the four-colorability proofs in sections 4.1.4 for d_1=3 triangular adjacency constructions between surrounding modules, and sections 4.4.3 and 4.4.4

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for C1~C20 plane configurations, only when the $d_1=2$ connecting endpoints are colored with the central color of the adjacent surrounding modules does a merging coloring conflict arise. In other situations, the endpoints can be merged into a color that is different from the central color of the adjacent surrounding modules. In Figure 54, at point ④, the endpoints $v_k^{(1)}$ (odd, ○) and $v_0^{(2)}$ (central, ○) of surrounding modules A and B share the same color, and similarly, the endpoints $v_k^{(2)}$ (even, ☆) and $v_0^{(1)}$ (central, ☆) of surrounding modules B and A are also the same color. Since the coloring conflicts arise at the endpoints $v_k^{(1)}$ and $v_k^{(2)}$ with their adjacent central endpoints of the surrounding modules, they cannot be directly merged according to section 4.1.3. Instead, the coloring of the connecting endpoints needs to be appropriately substituted. The same logic applies to points ⑤ and ⑥. The situation described involves adjacency conflicts between modules A and B, A and C, and B and C due to $d_1=2$ adjacent endpoints. This is the most complex case for triangular adjacency diagrams. To build upon this complexity, we construct the most complex layout of adjacent positions for all surrounding modules, i.e., the adjacency positions of the surrounding modules A (odd, even), B (odd, even), and C (even, even). Given that the color assignment for a surrounding module's odd and even endpoints has an exclusive relationship, the corresponding conflicting color assignments are the most complex. Moreover, the color conflict involving the odd and even endpoints of module A is two colors, with the minimum additional color for the odd and even endpoints of module B being an additional color on the basis of this. Together, this results in three colors. Since the central colors of all triangularly adjacent surrounding modules cannot be more than three, module C must also have a color conflict for its odd or even endpoints. Therefore, the constructed combination represents the most complex situation for the adjacency positions of all surrounding modules.

To sum up, the graph construction combinations for the II adjacency case between pairs of surrounding modules are as follows: the most number of adjacent endpoint pairs when $d_1=2$, the most complex merging coloring for adjacent endpoint pairs, and the most complex color cycling for adjacent positions within the same surrounding module. These conditions represent the most complex adjacency combination scenario, and through this scenario, four-colorability for plane configurations is proven. All other cases follow suit. To ensure that the tetrachromatic coloring for the II adjacency case between pairs of surrounding modules is proven, we apply the aforementioned rules to the adjacency scenarios in the already proven C1~C20 triangular adjacency situations. If the conditions continue to meet the requirement for four-colorability, then the four-colorability for the II adjacency case between pairs of surrounding modules is confirmed. This section will further discuss this proven case.

4.5.1.1. The C10 rules for construction prove four-colorability in the C60 scenario

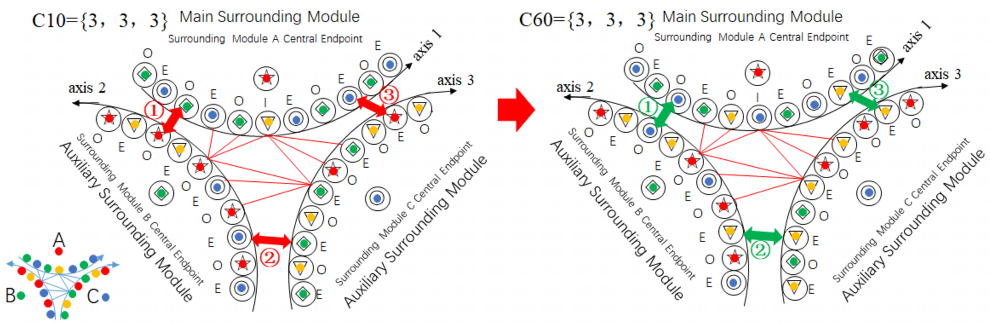

Figure 55: The C10 triangular adjacency pattern rules for constructing the coloring relationships between pairs in the C60 scenario

The diagram in Figure 55 illustrates the rule for a triangular adjacency situation, specifically focusing on the coloring relationships in a configuration of C60 molecules with two pairs of adjacent points. On the left side of the diagram, the primary structure involves a central module A, surrounded by auxiliary modules B and C, which are separated and merged at three pairs of points. There is a color conflict between these adjacent points and the center points of the adjacent modules. The inter-module adjacency positions represent the most complex scenario discussed previously. To adhere to the coloring rules as depicted in Figure 41, the adjacent points within each module could potentially be colored in a pattern of odd or even numbers, thus satisfying the triangular adjacency coloring requirements. However, in the case of modules A and B as shown in Figure 55, there is a conflict between odd and even colorings, precluding the use of an odd/even replacement coloring method. This necessitates that module C adopt this odd/even replacement pattern. Specifically, in the positive direction on axis 3 (odd marked with ▽; even marked with ☆), the replacement would be (odd, ☆; even, ▽), and in the negative direction on the

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same axis (odd, ▽; even, ▹), the replacement would be (odd, ▹; even, ▽). This replacement does not alter the sequential odd/even coloring pattern. The two points marked as ② and ③ in the diagram can be merged into a single point (▽). This can be achieved by merging the points between modules A and C, with the merged point of module A colored as (▽), differing from its original coloring (odd, ◇) and (even, ○). This effectively disrupts the sequential odd/even coloring rule. For the remaining point at location ①, which presents a coloring conflict, module A can be used to implement an odd/even replacement. This involves replacing the original coloring pattern (odd, ◇; even, ○) on axis 1 with (odd, ○; even, ◇). This allows the point at location ① to be merged into a single point (○) without disrupting the sequential odd/even coloring.

In the triangular adjacency for module A, the only necessary action is to swap the positions of the odd endpoints (○) with the initial color endpoints (▽). The initial color (▽) retains its status as an inserted color. This operation satisfies the coloring rules for adjacency points in a scenario with P=2n+1, specifically for odd-numbered loops. It ensures compliance with the adjacency conditions outlined in Figure 41 for adjacency situation 1. By implementing this coloring scheme, the C60 configuration with the structure {3, 3, 3} remains compatible with the four-coloring principle for planar graphs.

4.5.1.2. Display of the proof for four-colorability in planar graphs for scenarios C51 through C60 and C61 through C70

In the context of the already proven triangular adjacency scenarios for the C1-C20 cases, the above mentioned construction rules for the two-to-two adjacency II situations can be applied to the given conditions. The creation of C60 from C10 has already been demonstrated with a four-coloring proof. For the remaining combination scenarios, similar proofs can be constructed, though a detailed elaboration of these proofs is not provided in this article. For reference, see Appendices A-1 and A-2 which detail the coloring relationships for the two-to-two adjacent modules for the ranges C51-C60 and C61-C70 respectively. These appendices feature rule-based diagrams for C1-C10 and C11-C20, listing all combinations and demonstrating four-colorability. In each of the diagrams, the pairs of endpoints connected by double arrows at positions ①, ②, and ③ represent the combined endpoints of the two-to-two adjacent modules.

4.5.2. The four-colorability proof for adjacency case III of pairwise surrounding modules

In the adjacency situation III for surrounding modules, where the number of d_l=2 adjacent endpoints is k≥2, the adjacent endpoints can be categorized into continuous, spaced, and combinations of both. This section provides a planar four-coloring proof for these three scenarios.

4.5.2.1. The two-to-two adjacent endpoints in the surrounding modules with d_l=2 are continuous

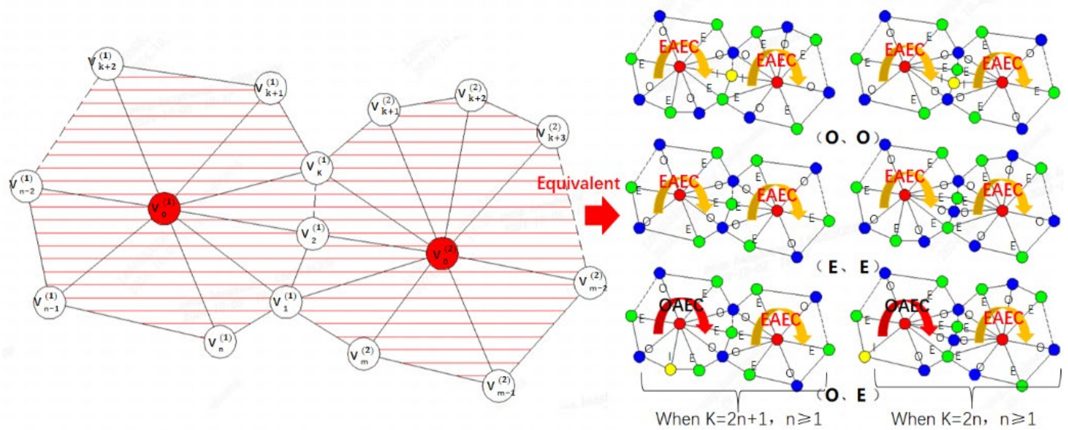

Figure 56: Construction and four-coloring proof for the adjacency of multiple consecutive d_l=2 endpoints in surrounding modules

As shown in Figure 56 on the left, the surrounding modules around the central endpoint $v_0^{(1)}$ and around the central endpoint $v_0^{(2)}$ are adjacent through continuous d_l=2 endpoints $v_1^{(1)}, v_2^{(1)},,, v_k^{(1)}$, respectively. According to the connectivity definition in this article, this forms a combined surrounding module, where the central endpoints $v_0^{(1)}$ and $v_0^{(2)}$ are colored the same color (red, ☆). The adjacent surrounding modules C_n and C_m can have three scenarios: (odd, odd), (odd, even), and (even, even). In

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the (odd, odd) scenario, the initial color I endpoints of the two odd loops can be constructed within the common endpoint of the connecting edges ($v_1^{(1)}, v_2^{(1)},,, v_k^{(1)}$, where k can be 2n+1 or 2n, with n ≥ 1). This results in the combined surrounding module sharing the equivalent of two even-adjacent endpoint circles (EAEC) outside the connecting edges in this case. In the (even, even) scenario, the combined surrounding module shares the equivalent of two even-adjacent endpoint circles (EAEC) outside the connecting edges in this case. In the (odd, even) scenario, the combined surrounding module shares the equivalent of one even-adjacent endpoint circle (EAEC) and one odd-adjacent endpoint circle (OAEC) outside the connecting edges in this case. As shown in the right side of Figure 56, the initial color I endpoints can be constructed according to the triangular adjacency zones of each surrounding module. The constructed combined surrounding module is equivalent to the various surrounding modules discussed in section 3 of this article, satisfying their own and adjacent four-coloring proofs.

4.5.2.2. The two-to-two adjacent endpoints in the surrounding modules with d_1=2 are spaced

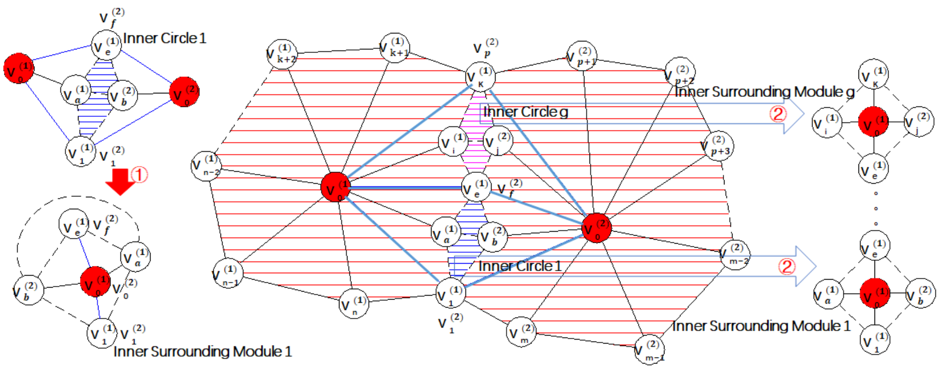

Figure 57: The construction and four-coloring proof for the adjacency of surrounding modules with multiple intervals d_1=2 endpoints

As shown in Figure 57, the surrounding modules surrounding the central endpoint $v_0^{(1)}$ and the surrounding modules surrounding the central endpoint $v_0^{(2)}$ are adjacent through non-consecutive d_1=2 endpoints $v_1^{(1)}$ / $v_1^{(2)},,, v_e^{(1)}$ / $v_f^{(2)},,, v_k^{(1)}$ / $v_p^{(2)}$. The internal loops ($v_1^{(1)}/v_1^{(2)},,, v_a^{(1)},,, v_e^{(1)}/v_f^{(2)},,, v_b^{(2)},,, v_1^{(1)}/v_1^{(2)}$) and ($v_e^{(1)}/v_f^{(2)},,, v_i^{(1)},,, v_k^{(1)}/v_p^{(2)},,, v_j^{(2)},,, v_e^{(1)}/v_f^{(2)}$) respectively form isolated loops, which can be filled in as shown in the blue and pink areas of Figure 57 (filling construction is the same as in Figure 26). Therefore, the surrounding modules surrounding the central endpoint $v_0^{(1)}$ and the surrounding modules surrounding the central endpoint $v_0^{(2)}$ can be regarded as a combined surrounding module, with the central endpoints $v_0^{(1)}$ and $v_0^{(2)}$ colored the same color (red, ☆). Based on the fact that the central endpoints of each internal loop are the same color and the connecting boundaries are the same with the number of endpoints ≥3, the center endpoint $v_0^{(1)}$ and $v_0^{(2)}$ of the internal loop 1 are colored the same. The connecting boundaries ($v_0^{(1)}, v_1^{(1)}/v_1^{(2)}$), ($v_0^{(2)}, v_1^{(1)}/v_1^{(2)}$) can be regarded as the same, and the connecting boundaries ($v_0^{(1)}, v_e^{(1)}/v_f^{(2)}$), ($v_0^{(2)}, v_e^{(1)}/v_f^{(2)}$) can be regarded as the same. Therefore, the center endpoint $v_0^{(1)}$ and $v_0^{(2)}$ can be graphically merged, and the connecting edges ($v_0^{(1)}, v_1^{(1)}/v_1^{(2)}$), ($v_0^{(2)}, v_1^{(1)}/v_1^{(2)}$) and ($v_0^{(1)}, v_e^{(1)}/v_f^{(2)}$), ($v_0^{(2)}, v_e^{(1)}/v_f^{(2)}$) can be graphically merged respectively. As shown in Figure 57 at point 1, the internal loop ($v_1^{(1)}/v_1^{(2)},,, v_a^{(1)},,, v_e^{(1)}/v_f^{(2)},,, v_b^{(2)},,, v_1^{(1)}/v_1^{(2)}$) is expanded outward to form a surrounding circle. The process of merging the expansion does not change the relative position and adjacency relationship of the planar graph, so the internal loop 1 can be graphically converted into an internal surrounding module 1. Based on the proof in previous sections, each surrounding module is planar and four-colorable. Therefore, as shown ② of the figure, the internal surrounding modules 1,..., internal surrounding module g are all planar and four-colorable.

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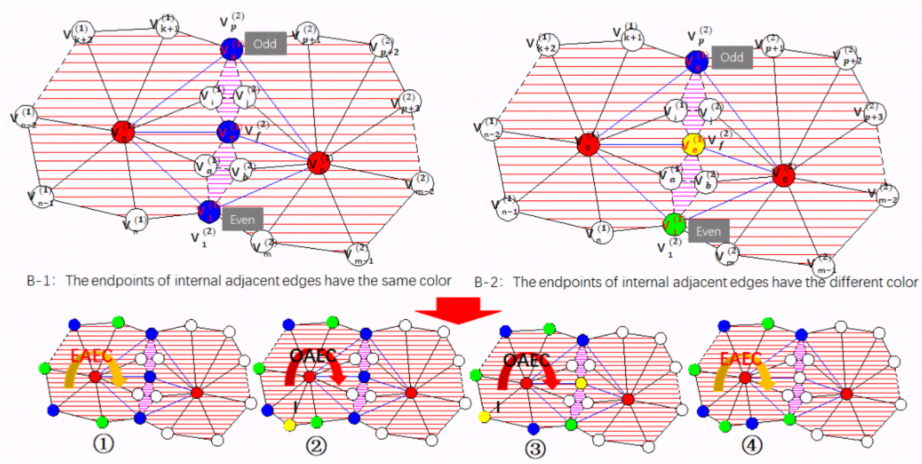

Figure 58: Continuation of construction and four-coloring proof for the adjacency of surrounding modules with multiple Intervals d_1=2 endpoints

Based on the coloring of external endpoints for each internal surrounding module, the adjacency endpoints $v_1^{(1)}/v_1^{(2)}, v_k^{(1)}/v_p^{(2)}$ can be categorized into two scenarios: same color and different colors, as illustrated in Figure 58. Without loss of generality, same color is assumed to be odd, and different colors are assumed to be odd and even respectively. As a result, the remaining connection endpoints outside the loops around $(v_{K+1}^{(1)}, v_{K+2}^{(1)},,, v_n^{(1)})$ can be either odd or even in number.In the odd case, under B-1, the external adjacency can be equivalently regarded as an even adjacency endpoint circle (EAEC) , the location marked with ① in the picture, and under B-2, as an odd adjacency endpoint circle (OAEC) , the location marked with ③ in the picture. In the even case, under B-1, it can be equivalently regarded as an OAEC, the location marked with ② in the picture, and under B-2, as an EAEC, the location marked with ④ in the picture. Moreover, the initial color I endpoint for the OAEC can be constructed based on the triangular adjacency regions for each equivalent encircling module, the yellow endpoints labeled I at positions ② and ④ in the diagram. The coloring construction for the equivalent surrounding modules around $v_0^{(2)}$ is the same as for $v_0^{(1)}$, and we do not repeat it here. The synthesized winding modules derived from this configuration can be equated to the various winding modules presented in section 3.

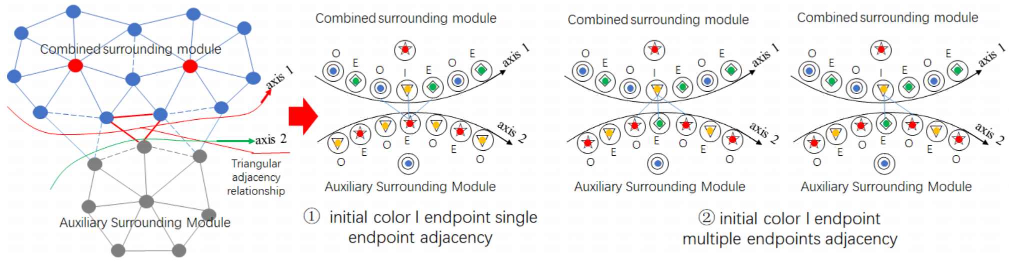

Figure 59: Proof of four-colorability for triangular adjacencies in combined surrounding modules

The adjacency relationship among the winding modules, as defined by the circle of adjacent nodes on the synthesized modules, translates into a two-dimensional axis adjacency in the manner depicted in Figure 59. The triangular adjacency within the synthesized modules is abstracted to the adjacency relationship between two-dimensional axes 1 and 2, akin to the adjacency in the auxiliary winding modules. In the context of triangular adjacency, there are two scenarios for the adjacency of the initial color I node within the synthesized module: single-endpoint adjacency and multi-endpoint adjacency. At position ① on the left, when it is a single endpoint, the auxiliary winding node is colored with the center color of the synthesized module (☆), following the coloring pattern (odd, ▽; even, ☆), which contrasts with the coloring of the synthesized module (odd, ○; even,◇). This pattern satisfies the four-colorable condition for planar graphs. At position ② on the right, when it is a multi-endpoint adjacency, the auxiliary winding node is colored as a combination of the center color (☆) and either an odd color (○) or an even color (◇). Given the interchangeable nature of odd and even, there are two categories of combinations depicted in the figure, both of which, in accordance with the adjacency properties of planar graphs, satisfy the four-colorable condition.

4.5.2.3. The combination of the above two situations

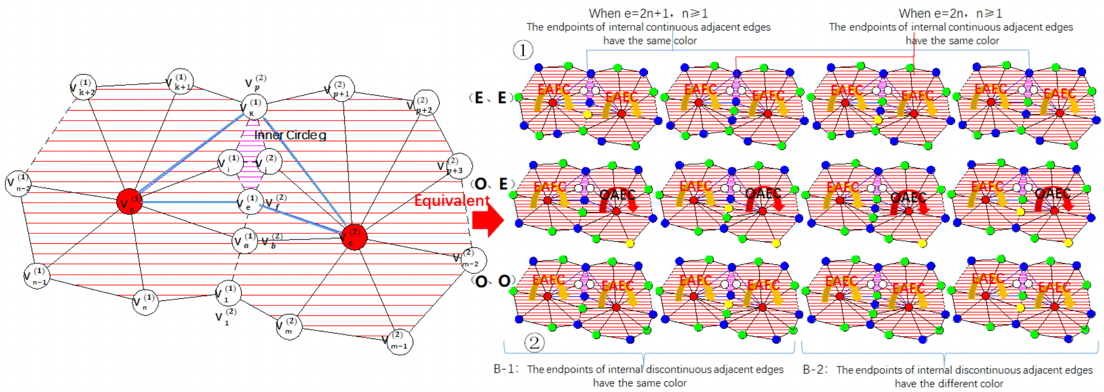

Figure 60: The four-colorability proof for the planar coloring construction under the first and second combination scenarios

As depicted in Figure 60, the configuration on the left is synthesized by combining two scenarios, namely, the module winding around the central endpoint $v_0^{(1)}$ and the module winding around the central endpoint $v_0^{(2)}$. These two modules are connected through consecutive endpoints $v_1^{(1)}/v_1^{(2)},,,v_a^{(1)}/v_b^{(2)},,,v_e^{(1)}/v_f^{(2)}$, and non-consecutive endpoints $v_e^{(1)}/v_f^{(2)},,,v_k^{(1)}/v_p^{(2)}$ The coloring structure resulting from this combination is illustrated on the right side of the figure. Vertically, the arrangement consists of the endpoints $v_{k+1}^{(1)}, v_{k+2}^{(1)},,,v_{n-1}^{(1)}, v_n^{(1)}$, and $v_{p+1}^{(2)}, v_{p+2}^{(2)},,,v_{m-1}^{(2)}, v_m^{(2)}$ with a combination of odd and even numbers, which correspond to the shared first and second scenarios. Horizontally, the first layer is for the combination of internal consecutive endpoints with counts e = 2n + 1 and e = 2n (where n is a positive integer), which represents the first scenario. The second layer is for the combination of non-consecutive internal endpoints B-1 and B-2, which represents the second scenario. All the resulting coloring combinations are listed on the right side of the figure. The combined winding modules can be equivalently regarded as various combinations of EAEC (Evenly Adjacent Endpoints Coloring) or OAEC (Oddly Adjacent Endpoints Coloring), satisfying the four-color theorem for themselves and their adjacent configurations.

In summary, we have conducted a proof of four-colorability for all adjacency cases of triangular adjacency among each pair of surrounding modules. This completes the proof of four-colorability for all combinations of individual surrounding modules, triangular adjacency among each pair of surrounding modules, and adjacencies between all pairs of surrounding modules.

4.6. Proof of four-colorability for the entire planar graph

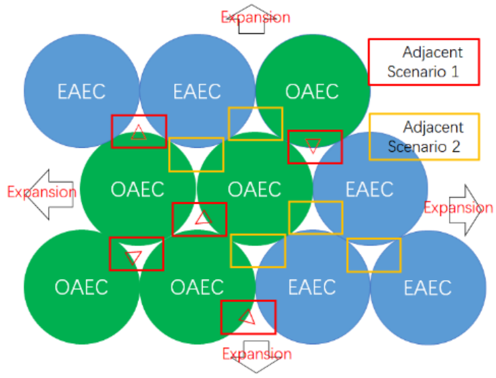

Figure 61: Demonstrates the growth of a planar diagram through the joining of various surrounding modules

Based on the proofs for adjacency cases 1 and 2 illustrated in Figure 41, as well as the decomposition relationships of the planar diagram into surrounding modules as presented in this paper, the entire planar diagram can be pieced together as shown in Figure 61, demonstrating the growth of the planar diagram through the joining of surrounding modules. The green surrounding modules are referred to as OAEC (short for Odd Adjacent Endpoint Circle), while the blue surrounding modules are designated as EAEC

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(short for Even Adjacent Endpoint Circle). Given that the number of potentially adjacent endpoints outside of an surrounding module is either odd or even, the surrounding module is therefore either an OAEC or an EAEC. The figure constructs the possible adjacency scenarios including (OAEC, OAEC, OAEC), (OAEC, EAEC, OAEC), (OAEC, EAEC, EAEC), and (EAEC, EAEC, EAEC). Hence, the entire planar diagram can be constructed in this manner.

Based on the interchange and rotation relationships of endpoint coloring in planar graphs, these conform to the adjacency models of encircling modules constructed in Figure 39: specifically, adjacency case 1 (highlighted by red rectangles in the figure, where the triangle symbol, ▽, within the rectangle indicates the module where the initial color, ▽, needs to be inserted during the pairing and coloring in the adjacency area. This corresponds to the color schematics for endpoints that can be adjoined externally to an odd number of circles satisfying P=2n+1, considering there is only one initial color endpoint, therefore each OAEC necessitates schematics creation at one place only) or adjacency case 2 (highlighted by yellow rectangles). This adjacency case has already been demonstrated to be four-colorable in this paper based on the aforementioned planar graph tiling construction, which can be continuously conducted. In doing so, the entire planar graph can be covered, thereby allowing the entire planar graph to be constructed to be four-colorable.

5. Conclusions

In this paper, based on graph theory and the law of logical thinking, in-depth analysis of planar graph construction and planar graph coloring problems, such as Table 2 planar graph four colorable proof steps, the planar graph four colorable proof.

Table 2: Planar graph four colorable proof steps

NO.	Proof steps
1	Construction of maximal planar graphs (2.3) and planar graph decomposition (2.4)
2	Prove that the surrounding module 1 is four-colorable (3.1) based on the number axis relation
3	Proof of four-colorability for each surrounding module based on graph decomposition (3.2)
4	Explanation of triangular adjacency relationships for each surrounding module and unification to the case of d_1=3 (4.1)
5	Simplification of adjacent coloring for encircling rings based on equivalence class partitioning (4.3)
6	Establishment of finite adjacent combinations based on d_1=3 and simplified adjacent coloring (4.4)
7	Verification of four-colorability for adjacent combinations (4.4.3-4.4.4)
8	The triangular adjacency combinations, initially proven for the case where d_1=3, are now extended and proven for scenarios where d_1=2. (4.4.5)
9	The proof has been extended from triangular adjacency to pairwise adjacency, encompassing the combination scenarios where d_1= 2 and d_1= 3. (4.5)
10	Proving four-colorability of planar graphs via the combination of OAEC and EAEC rules (4.6)

1) Construction of maximal planar graphs (2.3) and planar graph decomposition (2.4): To standardize the graph model for proving the planarity of graphs, internal adjacency edges are added in finite planar graphs to transform all endpoints into a triangular adjacency relationship. Based on this, the planar graph is uniformly decomposed into encircling sub-modules according to the degree of each endpoint, facilitating the proof of the four-color theorem for planar graphs.

2) Proving that the surrounding module 1 is four-colorable based on the number axis relation (3.1): based on the graph decomposition in step 1, decomposing the planar graph into different surrounding sub-modules, and proving that the surrounding module 1 is plane four-colorable by using the number axis mapping relation;

3) Four-color proofs for all surrounding modules based on graph decomposition construction (3.2): Building on the proof of the four-colorability of surrounding module 1, this step extends the proof to the four-colorability of all other surrounding modules that are constructed following the same decomposition process. This effectively proves the four-color theorem for the different surrounding sub-modules obtained from the decomposition of the planar graph.

4) Explanation and standardization of triangular adjacency for each surrounding module as d_1=3 (4.1): Due to the non-uniform distances between the centers of surrounding modules generated from the planar graph decomposition in step 1, local surrounding modules are connected with internal filling constructions. All scenarios are standardized to a triangular adjacency with d_1=3, further simplifying the combination cases for proving the four-color theorem of the planar graph.

5) Simplification of adjacent coloring for each encircling ring based on equivalence class division (4.3): The coloring of endpoints that can connect to the outside of each encircling ring is simplified by re-coloring according to the coloring patterns of odd rings and even rings. Even rings can be colored alternately in odd and even colors, while odd rings insert an initial color, which is assigned to the endpoint in the triangular adjacency area, standardizing the coloring of endpoints, and further simplifying the combination cases for proving the four-color theorem.

6) Establishment of a finite number of adjacency combinations based on d_1=3 and simplified adjacent coloring (4.4): Based on the regularized and simplified coloring rules of adjacency relationships, a three-dimensional number axis diagram model is abstracted to represent adjacency relationships. The endpoints of each surrounding module and their values on the corresponding number axis are taken and replaced with equivalent values, converting them into a finite number of value combinations, further simplifying the combination cases for proving the four-color theorem.

7) Separate proofs of triangular adjacency combination cases for planar four-colorability (4.4.3-4.4.4): Based on the combinations established in step 6, four examples are provided to prove their planar four-colorability. Other finite combinations are proven using the same method, demonstrating the planar four-colorability for all the established combinations.

8) Extension from d_1=3 to d_1=2 in the proven cases of triangular adjacency combinations (4.4.5): Based on the proofs for d_1=3 in step 7, the proof is expanded to cover the cases with d_1=2, proving that all constructed triangular adjacency cases satisfy the planar four-color theorem.

9) Proofs for combinations of d_1=2 and d_1=3 in two-by-two adjacency under triangular adjacency (4.5): Beyond the triangular adjacency, surrounding modules have pairwise adjacency outside the adjacency window. The center distances vary in combinations of d_1=2 and d_1=3, and a proof of the planar four-color theorem for these cases is provided.

10) Proving planar four-colorability through the rule-based combination of OAEC and EAEC (4.6): Based on the graph construction of adjacent surrounding modules in step 6 and the proofs of various adjacency relationships for planar four-colorability, this work employs the rule-based combination of OAEC (Ordered Adjacent Endpoint Construction) and EAEC (Equivalent Adjacent Endpoint Construction) for tiling. This enables the completion of the entire finite planar graph, thereby proving the four-color theorem for the entire planar graph.

As W.T. Tutte (1917-2002), the modern patriarch of graph theory, once remarked with profound insight, "The four-color problem is the tip of the iceberg, the point of the wedge, the first cry in the spring"[27]. Grounded in the logic of graph theory, this paper has undertaken to provide substantiated proofs concerning the four-colorability of planar graphs. It is our sincere aspiration that the validations presented herein will contribute meaningfully to the fields of mathematics and computer science.In line with Thomas's articulation in "An update on the four-color theorem," the proof of the four-color conjecture was not merely a victory in mathematics; it was a masterpiece that exemplified the confluence of logic, algorithms, and contemporary computer science [28]. This paper, inspired by such precedent, has been a journey of incessant exploration and innovation. Even when faced with challenges that may at first appear insurmountable, it is our firm belief that through tenacity and ingenuity, we will invariably discover methods and pathways to surmount them.

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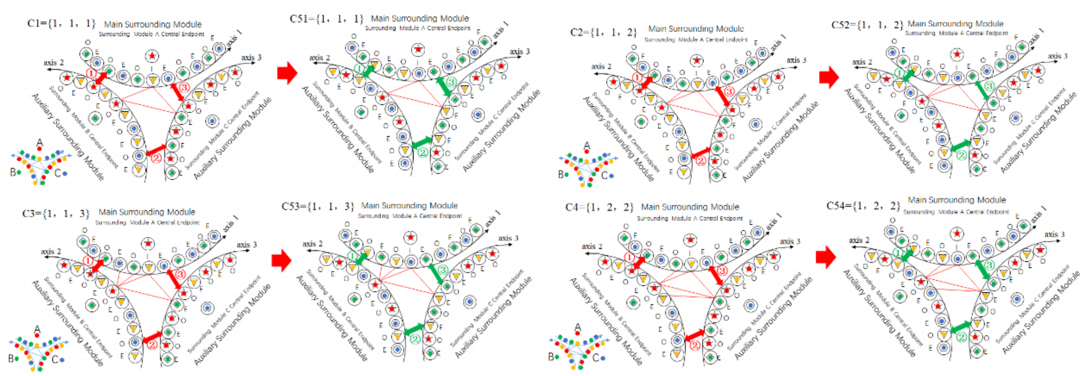
Appendix A

A-1. Display of the proof for four-colorability in planar graphs for scenarios C51 through C60



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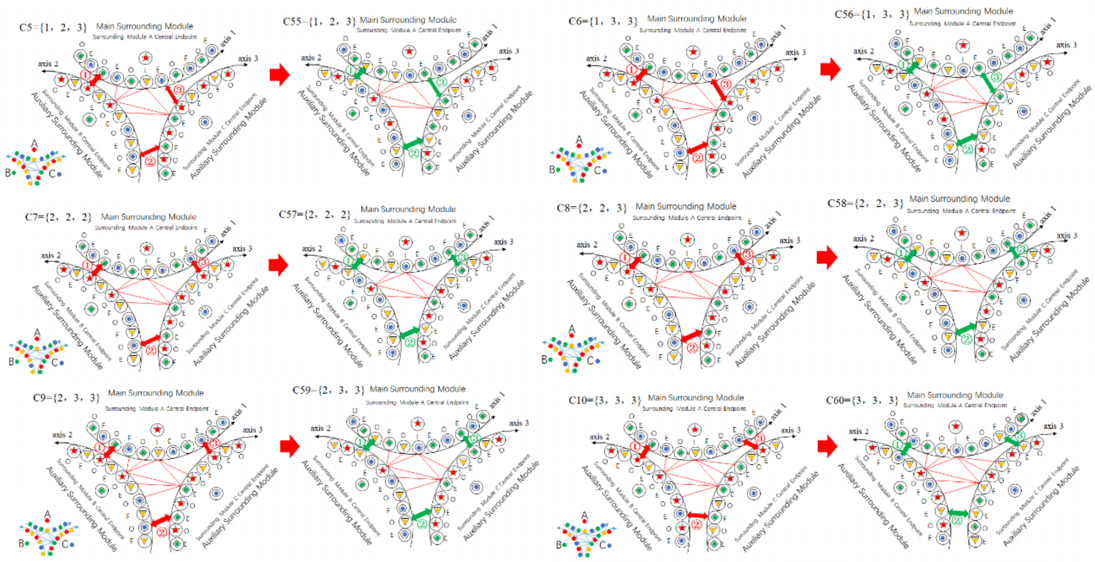

A-2. Display of the proof for four-colorability in planar graphs for scenarios C61 through C70