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Inheritance Properties of Proper Subspace

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ABSTRACT. *The topology, which is now a branch of geometry in modern mathematics, is now an important section. In this paper the local density, the weak density and the local weak density of topological spaces are investigated.*

KEYWORDS: *local density, weak density, local weak density, character.*

1. Introduction

We know from topology, for a topological property P, the question of whether a space has property P often arises. If the statement is true, the proof of the converse statement always quick and easy since the space is a subspace of if self. As a result, in a recent paper [1], the question of whether or not the space has property P knowing only that each proper subspace has property P was raised leading to the introduction and investigation of proper subspace inherited properties.

R.B. Beshimov [2] studies the notion of a weak separable space. Some counterexamples, which show a strictness of previous results of the author, are given. The weak separability of hyperspaces and function spaces is investigated.

We can show about weak density in [2], local density and local weak density in [3] some information and the theorems given below. Of course, these results are important for topologic spaces and in technology development.

2. Preliminaries

This section includes notation and a few important definitions.

Definition 2.1. *Let P be a property of topological spaces. If for each space for which each proper subspace has property P, the space has the property P, then property P is said to be a proper subspace inherited property* (PSIP) [2].

Definition 2.2. *The weak density of a topological space X is the smallest cardinal number $\tau \geq \aleph_0$ such that there is a π - base(π -network) in X coinciding with τ centered systems of* open *sets , i.e. there is a π - base B =*

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$\bigcup\{B_\alpha : \alpha \in A\}$, *where* B_α *is a centered system of open sets for each* $\alpha \in A$ *and* $|A| = \tau$ [4].

The weak density of a topological space X is denoted by $wd(X)$. If $wd(X) \le \aleph_0$ then we say that a topological space X is weakly separable.

Definition 2.3. *We say that a topological space* X *is locally separable at a point* $x \in X$ *if* x *has separable neighborhood.*

A topological space is locally separable if it is separable at each point $x \in X$ [3].

Definition 2.4. *We say that a topological space* X *is locally* τ *-dense at a point* $x \in X$ *if* τ *is the smallest cardinal number such that* x *has* τ *-dense neighborhood in* X .

The local density at a point x is denoted by $ld(x)$ .The local density of a space X is defined as the supremum of all numbers $ld(x)$ for $x \in X$; this cardinal number is denoted by $ld(X)$ [3].

If $ld(x) \le \aleph_0$ for a space X , we say that X is locally separable.

Definition 2.5. *A topological space is locally weakly* τ *-dense at a point* $x \in X$ *if* τ *is the smallest cardinal number such that* x *has a neighborhood of weak density* τ *in* X .

The local weak density at a point x is denoted by $lwd(X)$.

A topological space X is called locally weakly τ -dense if it locally weakly τ -dense at each point $x \in X$. The local weak density of a topological space X is defined with following way: $lwd(X) = \sup\{lwd(x) : x \in X\}$ [3].

Definition 2.6. *The pseudo-character of a point* x *in a* T_1*-space* X *is defined as the smallest cardinal number of the form* $|u|$, *where* u *is a family of open subsets of* X *such that* $\cap u = \{x\}$; *this cardinal number is denoted by* $\psi(x, X)$. *The pseudo-character of a* T_1 *-space* X *is defined as the supremum of all numbers* $\psi(x, X)$ *for* $x \in X$; *this cardinal number is denoted by* $\psi(X)$ [5].

Definition 2.7. *A family* $\mathrm{B}(x)$ *of neighborhoods of* x *is called a base for a topological space* (X, τ) *at the point* x *if for any neighborhood* V *of* x *there*

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exists a $U \in \mathrm{B}(x)$ *such that* $x \in U \subset V$ *. One can easily check that if* B *is a base for* (X,τ) *at the point* x *. On the other hand, if for every* $x \in X$ *a base* $\mathrm{B}(x)$ *for* (X,τ) *at the point* x *is given, then the union* $\mathrm{B} = \bigcup_{x \in X} \mathrm{B}(x)$ *is a base for the space* (X,τ).

The character of a point x in a topological space (X,τ) is defined as the smallest cardinal number of the form $|\mathrm{B}(x)|$, where $\mathrm{B}(x)$ is a base for (X,τ) at the point x; this cardinal number is denoted by $\chi(x,(X,\tau))$. The character of a topologic space (X,τ) is defined as the supremum of all numbers $\chi(x,(X,\tau))$ for $x \in X$; this cardinal number is denoted by $\chi(x,(X,\tau))$ or simply $\chi(X)$.

$\chi(X) \le \aleph_0$-first countable

$w(X) \le \aleph_0$-second-countable

If $\chi(x,(X,\tau)) \le \aleph_0$, then we say that the space (X,τ) satisfies the first axiom of countability or is first-countable; this means that at every point x of X there exists a countable base[5].

Theorem 2.1. For every topological space X we have $d(X) \le w(X)$ [5].

Corollary 2.1. Every second-countable space is separable[5].

Definition 2.8. *If* $p \in X$, *a local* π *-base of* p *in* X *is a family* $\mathfrak{A} \subset \tau(X) \setminus \{\varnothing\}$ *such that every neighborhood of* p *contains a member of* $\mathfrak{A}$.

$$\pi\chi(p,X) = min\{|\mathfrak{A}| : \mathfrak{A}\ a\ local\ \pi - base\ of\ \ p\ in\ X\}$$

is the π *- character of p in* X .

$$\pi\chi(X) = \sup\{\pi\chi(p,X) : p \in X\}$$

is the π *-character of* X [6].

3. Main results

In this section, we prove that proper subspace some properties.

If for some $x \in X$ and an open set $U \subset X$ we have $x \in U$, we say that U is a neighborhood of x.

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Theorem 3.1. If $wd(Y) \leq \tau$ for any proper subspace Y of a topological space X, then $wd(X) \leq \tau$.

Proof. Let X be a topological space, such that $wd(Y) \leq \tau$, for every proper subspace Y. Let $x \in X$ and $Y = X \setminus \{x\}$. Then there is a π –network $\beta = \bigcup_{\alpha \in A} \beta_\alpha$ in Y, coinciding with τ centered systems, where $|A| \leq \tau$. Put $\beta_x = (E \subset X : x \in E)$ and consider family $\overline{\beta} = \beta \cup \beta_x$. Then $\overline{\beta}$ is a π –network in X and, obviously, coincides with τ centered systems. Indeed, let U be an arbitrary open set in X. If $x \in U$ then $U \in \beta_x \subset \overline{\beta}$ and $U \subset U$. Now consider $x \notin U$. Then U is open in Y, since $U \cap Y = U$. Since β is a π -network in Y, there is an element $E \in \beta \subset \overline{\beta}$ such that $E \subset U$. Therefore $\overline{\beta}$ is a π -network in X, coinciding with τ centered systems. Hence $wd(X) \leq \tau$. Theorem 3.1. is proved.

Corollary 3.1. Weakly separability is a PSIP.

Theorem 3.2. If $ld(Y) \leq \tau$ for any proper subspace Y of a topological space X, then $ld(X) \leq \tau$.

Proof. Let (X, τ) be a topological space. Let $x \in X$ and $Y = X \setminus \{x\}$. Then $ld(Y) \leq \tau$. For an arbitrary point $y \in Y$ there is a neighborhood $O_Y y$ in Y such that $d(O_Y y) \leq \tau$. Put $Oy = O_Y y$, if $O_Y y \in \tau$ and $Oy = O_Y y \cup \{x\}$, if $O_Y y \notin \tau$. Then, it is clear that Oy is a neighborhood of y and $d(Oy) = d(O_Y y) \leq \tau$. Therefore $ld(y) \leq \tau$. Now consider the point x. If there is not neighborhood of x except of X, then one point set $\{x\}$ is dense in X and $d(X) = 1 < \tau$. Now suppose that there is a neighborhood U of x such that $U \neq X$. Then $ld(U) \leq \tau$, since U is a proper subspace of X. Consequently, there exists a neighborhood Ox of x in U such that $d(Ox) \leq \tau$. But U is open in X, and therefore Ox is also open in X. So, we found a neighborhood Ox for x in X such that $d(Ox) \leq \tau$. Hence $ld(X) \leq \tau$. Theorem 3.2 is proved.

Corollary 3.2. Locally separability is a PSIP.

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Theorem 3.3. If $lwd(Y) \le \tau$ for any proper subspace Y of a topological space X , then, $lwd(X) \le \tau$.

Proof. Let (X, τ) be a topological space. Let $x \in X$ and $Y = X \setminus \{x\}$. Then, $ld(Y) \le \tau$. For an arbitrary point $y \in Y$ there is a neighborhood $O_Y y$ in Y that $wd(Y) \le \tau$. Put, $Oy = O_Y y$, if $O_Y y \in \tau$ and $Oy = O_Y y \cup \{x\}$, if $O_Y y \notin \tau$. Then, it is clear that Oy is a neighborhood of y and, $wd(Oy) = wd(O_Y y) \le \tau$ .Hence $lwd(y) \le \tau$. Now let us check neighborhoods of the point x . If there is not neighborhood of x except of X , then one point set $\{x\}$ in dense in X and $wd(X) = 1 < \tau$. Now suppose that there is a neighborhood U of x such that, $U \ne X$. Then, $lwd(U) \le \tau$, since U is a proper subspace of X . Consequently, there exists a neighborhood Ox of x in U such that, $wd(Ox) \le \tau$. But U is open in X , and therefore Ox is also open in X . So, we found a neighborhood Ox for x in X such that, $wd(Ox) \le \tau$. Hence, $lwd(X) \le \tau$. Theorem 3.3 is proved.

Corollary 3.3. Locally weakly separability is a PSIP.

Theorem 3.4. If $\psi(Y) \le \tau$ for every proper subspace Y of a T_1-space X , then $\psi(X) \le \tau$.

Proof. Let $x \in X$. Since the space X is T_1 , there is a open subset U in X such that, $x \in U$. We have $\psi(U) \le \tau$ and this implies that there is a system $\{U_\alpha : \alpha \in A\}$ of open sets in U such that $\bigcap_{\alpha \in A} U_\alpha = \{x\}$ and, $|A| \le \tau$. Since U is open in X , every U_α is open in X $\bigcap_{\alpha \in A} U_\alpha = \{x\}$ .Therefore we have $\psi(x, X) \le \tau$. and –the point x being arbitrary- $\psi(X) \le \tau$ .Theorem 3.4 is proved.

Theorem 3.5. First countable is proper subspace inherited property.

Following theorem is generalization of theorem 3.1.

Theorem 3.6. If $\chi(Y) \le \tau$ for any proper subspace of a space X then, $\chi(X) \le \tau$.

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Proof. Let X be a space such that every proper subspace $Y \subset X$ $\chi(Y) \le \tau$. Let $x \in X$

Without loss of generality, we can assume that there exists a neighborhood U of x, which is a proper subset of X. Then, $\chi(U) \le \tau$. Let β_x be a base at x in U with $|\beta_x| \le \tau$. Since each element of β_x is open U and U open in X, each element of β_x is open in X. Therefore β_x is a base at the point x in X and $|\beta_x| \le \tau$. Hence $\chi(X) \le \tau$. Theorem 3.6 is proved.

Theorem 3.7. If $\pi\chi(Y) \le \tau$ for any proper subspace Y of a topological space X, then $\pi\chi(X) \le \tau$.

Proof. Let X be a space such that every proper subspace $Y \subset X$ $\pi\chi(Y) \le \tau$. Let $x \in X$. We can assume that there exists a neighborhood U of x, which is a proper subset of X. The $\pi\chi(U) \le \tau$. Let β_x be a π-base at x in U with $|\beta_x| \le \tau$. Since each element of β_x is open in U and U open in X, each element of β_x is open in X. Therefore, β_x is a base at the point x in X and $|\beta_x| \le \tau$. Hence $\pi\chi(X) \le \tau$. Theorem 3.7 is proved.

Example. Let X be the discrete space of cardinality c. It is obvious that ld(X) = lwd(X) = 1 < $\aleph_0$, i.e. it is locally separable and locally weakly separable [7].

4. Conclusion

In this article, we have presented some facts of how topological methods can be applied to study data sets. We have seen above, weakly separability, locally separability and locally weakly separability also PSIP. We will carry out our basic scientific research using these properties in our subsequent work.

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