A Prospect Theory Based Probabilistic Interval-Valued Hesitant Fuzzy Sets with Interval Reference Point and Incomplete Weight Information for Multi-Criteria Decision Making

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ABSTRACT. *In this paper, we introduce probabilistic interval-valued hesitant fuzzy sets (PIVHFS) and probabilistic interval-valued hesitant fuzzy elements (PIVHFE), then define correlation operations. At the same time, the hybrid distance measures of PIVHFE and interval fuzzy number are defined, and the hybrid distance measures of two PIVHFEs are also defined. Considering the psychological behavior of decision makers, we use interval fuzzy number decision makers' psychological interval reference points to calculate the gains and losses of evaluation value. Furthermore, based on incomplete weight information, the grey relational analysis (GRA) method is extended to calculate the optimal weight. Then, based on the traditional TOPSIS method, the decision-making method is proposed. Finally, the proposed method is applied to an example, and the sensitivity analysis and stability analysis of the proposed method are also discussed.*

KEYWORDS: *Probabilistic interval-valued hesitant fuzzy sets; Prospect theory; Gains and losses; Incomplete weight information; Topsis method*

1. Introduction

With the rapid development of society, economy and technology, there are a lot of multi-criteria decision making (MCDM) problems in many fields. Before the development of fuzzy set theory, decision makers used precise numbers to evaluate the problem attributes when studying MCDM problems. For example, when evaluating a swimmer, one of the criteria is the swimming speed, which can be represented by a real numbers. However, in practical problems, many cases cannot be evaluated with precise figures. For example, when a company selects talents, one of the selection criteria is the ability to adapt to the changing situation, which cannot be expressed with real numbers. The fuzzy set theory has effectively solved this problem.

As an extension of fuzzy set theory, the hesitant fuzzy set is widely used in MCDM problems. In 2010, Torra [17] proposed the concept of hesitant fuzzy set (HFS), which allows the degree to which an object belongs to the fuzzy set to be given in the form of a set of multiple possible values, and can effectively describe the uncertainty in decision making. Sometimes it is very difficult for decision makers to give an accurate number for evaluation, so Chen et al. [18] proposed the interval-valued hesitant fuzzy set (IVHFS) in 2013. In 2014, zhu [19] proposed the probabilistic hesitant fuzzy set (P-HFS), which gives an accurate evaluation considering that decision makers are not fully confident.

The TOPSIS method was first proposed by Hwang and Yoon [26] in 1981. This method is often used in MCDM problems and has become a classic method. In recent years, based on the traditional TOPSIS method, many improved and expanded methods have emerged one after another. In 2014, Zhang and Xu [20], on the basis of Pythagorean fuzzy set, determined the positive ideal solution and the negative ideal solution by using the score function comparison method, and thus proposed the TOPSIS expansion method. Xu and Liang [21] selected the set with the maximum membership degree and the minimum non-membership degree as the positive ideal solution and the set with the minimum membership degree and the maximum non-membership degree as the negative ideal solution on the basis of Pythagorean fuzzy set in 2017.

Since Kahneman and Tversky [1] proposed prospect theory (PT) in 1979, behavioral decision theory has developed rapidly. In 1982 and 1985, Bell [22-23] proposed the regret theory and disappointment theory, respectively, in 1992, Kahneman and Tversky [24] proposed the cumulative PT. Zhou and Li [25] gave the calculation method of the gains and losses of the benefit-type attribute, cost-type attribute, respectively, then calculated the behavioral weight according to the gains, losses and prospect value.

The rest of the paper is organized as follows. In section 2, we introduce PIVHFS and PIVHFE, then give their properties, and define correlation operations. And the generalized distance measure between PIVHFE and interval fuzzy numbers and generalized distance measure of two PIVHFEs are defined. In section 3, we discuss the position relation between the evaluation value and the interval reference point and the calculation method of the gains and losses of the two type attributes. In section 4, based on incomplete weight information, an extension method of GRA is proposed to calculate the optimal weight vector. In section 5, on the basis of the traditional TOPSIS method, puts forward a decision-making method. In section 6, the proposed method is applied to an example. The conclusion is given in section 7.

2. Basic Conception

2.1 Prospect Theory

Prospect theory [1] is a theory that describes and predicts people's behavior inconsistent with the traditional expected value theory and expected utility theory in

the risk decision making process which was proposed by Daniel Kahneman and Amos Tversky in 1979. Since prospect theory was put forward, there have been many literatures about prospect theory. In 1991, there was a paper on the application of prospect theory to riskless decision making [2]. Prospect theory is divided into two different processes, one is the calculation of gains and losses, the other is the calculation of prospect value. The magnitude of gains and losses is relative to the reference point, and the prospect value is obtained by combining the value function of gains and losses. The highest prospect value will be selected. According to literature [1], the value function of prospect theory is expressed as follows:

$$v(x) = \begin{cases} x^{\alpha}, & x \geq 0 \\ -\lambda(-x)^{\beta}, & x < 0 \end{cases} \qquad (1)$$

Where x represents gains or losses, $x \geq 0$ and $x < 0$ are gains and losses, respectively. α and β are the relevant parameters of gains and losses respectively, with $\alpha, \beta \in [0,1]$. λ is the risk aversion parameter and $\lambda > 1$. The larger λ means that people are more sensitive to losses than gains. The values of α, β and λ are determined by a series of experiments [3-6]. According to the determined values of α, β and λ, a prospect value function as shown in Figure 1 with convex and concove S-shapes value function can be given for gains and losses, respectively. To simplify the calculation, let $\alpha=\beta=0.88$, $\lambda=2.25$.
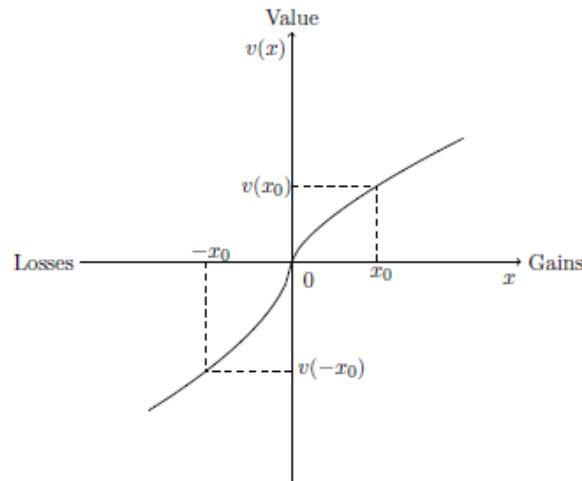

Figure.1 an s-Shapes Value Function of Prospect Theory

2.2 Probabilistic Interval-Valued Hesitant Fuzzy Set

Definition 1.[7] Let X be the reference set, and PIVHFS on X is a function that returns a random variable that is valued on the subset of [0,1] when applied to X. If the random variable in PIVHFSs is discrete, a PIVHFS can be written as follows:

$$H = \left\{ \langle x_i, h(x_i) \rangle \big| x_i \in X \right\} \tag{2}$$

where $h(x_i) = \left\{ \gamma_i(p_{\gamma_i}) \big| \gamma_i \in h(x_i) \right\}$ is a set of real numbers, $\gamma_i = \left[\gamma_i^L, \gamma_i^U \right]$ is interval-valued membership degree of $x \in X$ for set H with $\gamma_i^L = \inf \gamma_i$ and $\gamma_i^U = \sup \gamma_i$ represent the lower and supper limits of γ_i, respectively, p_{γ_i} is the probability of the possible membership degree γ_i with $\sum_{\gamma_i \in h(x_i)} p_{\gamma_i} = 1$ and $p_{\gamma_i} \in [0,1]$. For convenience, we call h a probabilistic interval-valued hesitant fuzzy element (PIVHFE). For the convenience of calculation, the elements in PIVHFE are sorted in ascending order.

2.3 The Normalization of a Set of Pivhfe

There are many studies on the normalization of hesitant fuzzy sets in literature, among which Zhang et al. [8] have studied the normalization process of probabilistic hesitant fuzzy sets. There are two main purposes of normalization: (1) all PIVHFEs have complete probability information; (2) all PIVHFEs have the same length. Now the following definition of the associated PIVHFE can be obtained. In order to achieve the second purpose of normalization, a new adjustment method is proposed in reference [9]. And similarly we can extend this method to PIVHFEs.

Let $h_1 = \left\{ \gamma_1^{(k)}(p_1^{(k)}) \big| k = 1, 2, \cdots, K_1 \right\}$ and $h_2 = \left\{ \gamma_2^{(k)}(p_2^{(k)}) \big| k = 1, 2, \cdots, K_2 \right\}$ be two normalized PIVHFEs. Suppose that the adjusted probabilistic distributions of h_1 and h_2 are the same as $P = \left(p^{*(1)}, p^{*(2)}, \cdots, p^{*(K)} \right)^T$ with $\sum_{k=1}^{K} p^{*(k)} = 1$. Then the adjusted PIVHFEs are $h_1^* = \left\{ \gamma_1^{*(k)}(p^{*(k)}) \big| k = 1, 2, \cdots, K \right\}$ and $h_2^* = \left\{ \gamma_2^{*(k)}(p^{*(k)}) \big| k = 1, 2, \cdots, K \right\}$. The details of the adjusting process, as shown in Example 1 and Figure 2.

Example 1. For two PIVHFEs $h_1 = \{[0.1,0.3](0.4),[0.4,0.6](0.6)\}$ and

$h_2 = \{[0.1,0.2](0.3),[0.3,0.5](0.5),[0.6,0.7](0.2)\}$, the adjusted

PIVHFEs are $h_1^* =$

$\{[0.1,0.3](0.3),[0.1,0.3](0.1),[0.4,0.6](0.4),[0.4,0.6](0.2)\}$ and

$h_2^* = \{[0.1,0.2](0.3),\ [0.3,0.5](0.1),[0.3,0.5](0.4),[0.6,0.7](0.2)\}$.
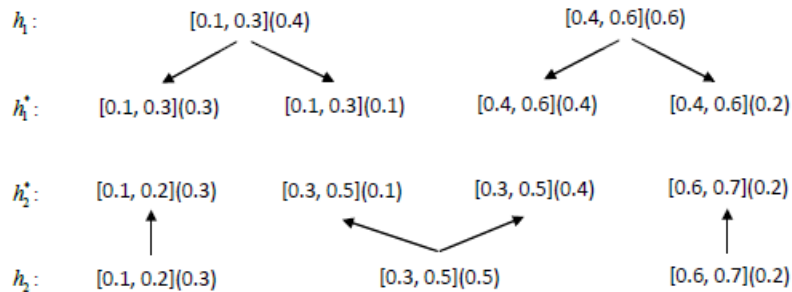
The calculation processes are shown in Figure 2.

Fig.2 : the Process of Adjusting the Pivhfes: h_1 *and* h_2

2.4 Some Basic Operations of the Adjusted Pivhfes

The new operations of PIVHFEs are defined as follows:

Definition 2. For two adjusted PIVHFEs $h_1^* = \{\gamma_1^{*(k)}(p^{*(k)})|k=1,2,\cdots,K\}$

and $h_2^* = \{\gamma_2^{*(k)}(p^{*(k)})|k=1,2,\cdots,K\}$, and $\lambda_1 > 0$, then

$(1)\left(h_1^*\right)^c = \bigcup_{k=1,2,\cdots,K}\left\{\left[1-\gamma_1^{*(k)U},1-\gamma_1^{*(k)L}\right](p^{*(k)})\right\}$;

(2)

$h_1^* \cap h_2^* = \bigcup_{k=1,2,\cdots,K}\left\{\left[\min\left(\gamma_1^{*(k)L},\gamma_2^{*(k)L}\right),\min\left(\gamma_1^{*(k)U},\gamma_2^{*(k)U}\right)\right](p^{*(k)})\right\}$;

(3)

$h_1^* \cup h_2^* = \bigcup_{k=1,2,\cdots,K}\left\{\left[\max\left(\gamma_1^{*(k)L},\gamma_2^{*(k)L}\right),\max\left(\gamma_1^{*(k)U},\gamma_2^{*(k)U}\right)\right](p^{*(k)})\right\}$;

$$(4)\left(h_1^*\right)^{\lambda_1} = \bigcup_{k=1,2,\cdots,K}\left\{\left[\left(\gamma_1^{*(k)L}\right)^{\lambda_1},\left(\gamma_1^{*(k)U}\right)^{\lambda_1}\right]\left(p^{*(k)}\right)\right\};$$

$$(5)\,\lambda_1 h_1^* = \bigcup_{k=1,2,\cdots,K}\left\{\left[1-\left(1-\gamma_1^{*(k)L}\right)^{\lambda_1},1-\left(1-\gamma_1^{*(k)U}\right)^{\lambda_1}\right]\left(p^{*(k)}\right)\right\}.$$

2.5 The Correlation Distance Measure of Pivhfes

In TOPSIS [10], VIKOR [11] methods, the distance between the alternative and the optimal solution or the worst solution is commonly used to measure the deviation, so as to obtain the decision result. For the distance measure between PIVHFE and the interval fuzzy number, if the element of PIVHFE appears in the interval fuzzy number, the distance measure between the element and the interval fuzzy number is 0; otherwise, the distance measure is expressed as the hamming distance between the element and the interval fuzzy number. So, the distance between the PIVHFE and interval fuzzy number is defined as follows.

Definition 3. Let $h=\left\{\gamma^{(k)}(p^{(k)})\big|k=1,2,\cdots,K\right\}$ be a PIVHFE and $e=\left[e^L,e^U\right]$ be an interval-valued fuzzy number, where e^L is the lower limit of membership degree and e^U is the upper limit of membership degree. Then the weighted hybrid probabilistic interval-valued hesitant fuzzy reference distance measure as follow:

$$d\left(h,e\right)=a\left(\sum_{k=1}^{K}p^{(k)}d\left(\gamma^{(k)},e\right)^q\right)^{\frac{1}{q}}+(1-a)\left(\max_{k=1}^{K}p^{(k)}d\left(\gamma^{(k)},e\right)^q\right)^{\frac{1}{q}} \qquad (3)$$

where $0\le a\le 1$ and $q>0$,

$$d\left(\gamma^{(k)},e\right)=\begin{cases}\dfrac{1}{2}\left(\left|\gamma^{(k)L}-e^L\right|+\left|\gamma^{(k)U}-e^U\right|\right),&\gamma^{(k)}\not\subset\left[e^L,e^U\right]\\[2mm]0&,\ \gamma^{(k)}\subset\left[e^L,e^U\right]\end{cases},$$

a is a parameter for determining the linear combination.

The distance measure between two PIVHFEs will be defined below.

Definition 4. Let $h_1=\left\{\gamma_1^{(k)}(p_1^{(k)})\big|k=1,2,\cdots,K_1\right\}$ and $h_2=\left\{\gamma_2^{(k)}(p_2^{(k)})\big|k=1,2,\cdots,K_2\right\}$ be two normalized PIVHFEs, suppose the

adjusted forms are $h_1^* = \left\{ \gamma_1^{*(k)}(p^{*(k)}) \middle| k = 1, 2, \cdots, K \right\}$ and

$h_2^* = \left\{ \gamma_2^{*(k)}(p^{*(k)}) \middle| k = 1, 2, \cdots, K \right\}$. The weighted hybrid generalized distance

between h_1 and h_2 can be defined as:

$$d\left(h_1, h_2\right) = a\left(\sum_{k=1}^{K} p^{*(k)} d\left(\gamma_1^{*(k)}, \gamma_2^{*(k)}\right)^q\right)^{\frac{1}{q}} + \left(1-a\right)\left(\max_{k=1}^{K} p^{*(k)} d\left(\gamma_1^{*(k)}, \gamma_2^{*(k)}\right)^q\right)^{\frac{1}{q}} \quad (4)$$

where $0 \le a \le 1$, and $q > 0$,

$$d\left(\gamma_1^{*(k)}, \gamma_2^{*(k)}\right) = \frac{1}{2}\left(\left|\gamma_1^{*(k)L} - \gamma_2^{*(k)L}\right| + \left|\gamma_1^{*(k)U} - \gamma_2^{*(k)U}\right|\right)$$, a is parameters for

determining the linear combination.

3. Calculation of Gains and Losses

In prospect theory, the final prospect value is obtained according to gains and losses. In this section, we use the weighted hybrid probabilistic interval-valued hesitant fuzzy reference distance measure of attribute value and interval reference point to measure gains and losses. This section mainly discusses the position relation between interval reference point and attribute value in the probabilistic interval-valued hesitant fuzzy environment, and the calculation of gains and losses. In order to facilitate the calculation of gains and losses, a new definition is given in this paper before the discussion as follows.

Definition 5. Let $h = \left\{ \gamma^{(k)}(p^{(k)}) \middle| k = 1, 2, \cdots, K \right\}$ be a PIVHFE and I be a

subset of $\left\{1, 2, \cdots, K\right\}$, then a subset $\tilde{h}$ of PIVHFE h is called SPIVHFE and

obviously $\tilde{h}$ is also a PIVHFE:

$$\tilde{h} = \left\{ \gamma^{(k)}(p^{(k)}) \middle| k \in I \right\} \quad (5)$$

Let $e_j = \left[e_j^L, e_j^U\right](j = 1, 2, \cdots, n)$ be an interval reference point, where e_j

is a interval fuzzy number and stands for decision maker's psychological interval

reference point with respect to the jth attribute. Let

$h_{ij} = \left\{ \gamma_{ij}^{(k)}(p_{ij}^{(k)}) \middle| k = 1, 2, \cdots, K \right\}(i = 1, 2, \cdots m; j = 1, 2, \cdots n)$ be an attribute

values with $\gamma_{ij}^{(k)} = \left[\gamma_{ij}^{(k)L}, \gamma_{ij}^{(k)U}\right]$ and $\gamma_{ij}^{(k)U} > \gamma_{ij}^{(k)L}$, where h_{ij} is a PIVHFE

and stands for jth attribute value with respect to the ith alternative. Here, let

$$\max_{k}\left\{\gamma_{ij}^{(k)}\right\} = \left[\max_{k}\left\{\gamma_{ij}^{(k)L}\right\}, \max_{k}\left\{\gamma_{ij}^{(k)U}\right\}\right] = \overline{\gamma}_{ij} = \left[\overline{\gamma}_{ij}^{L}, \overline{\gamma}_{ij}^{U}\right]$$ and

$$\min_{k}\left\{\gamma_{ij}^{(k)}\right\} = \left[\min_{k}\left\{\gamma_{ij}^{(k)L}\right\}, \min_{k}\left\{\gamma_{ij}^{(k)U}\right\}\right] = \underline{\gamma}_{ij} = \left[\underline{\gamma}_{ij}^{L}, \underline{\gamma}_{ij}^{U}\right]$$. To clearly

express the relationship between h_{ij} and e_j , their relative positions on the coordinate axis are given as shown in Table 1.

As shown in Table 1, the position relationship between h_{ij} and e_j can be divided into the following categories. The following is just a discussion of benefit-type attributes.

(1) For case 1, obviously, there is no gain for decision making, and since

$$\overline{\gamma}_{ij}^{U} < e_{j}^{U}$$, then

$$G_{ij} = 0, L_{ij} = -d\left(h_{ij}, e_{j}\right), \left(i = 1, 2, \cdots m; j = 1, 2, \cdots n\right) \tag{6}$$

(2) For case 2, since $e_{j}^{L} < \underline{\gamma}_{ij}^{L}$, then

$$G_{ij} = -d\left(h_{ij}, e_{j}\right), L_{ij} = 0, \left(i = 1, 2, \cdots m; j = 1, 2, \cdots n\right) \tag{7}$$

(3) For case 3, since $\gamma_{ij}^{(k)} = \left[\gamma_{ij}^{(k)L}, \gamma_{ij}^{(k)U}\right] \subset \left[e_{j}^{L}, e_{j}^{U}\right]$ for all k , then

$$G_{ij} = 0, L_{ij} = 0, \left(i = 1, 2, \cdots m; j = 1, 2, \cdots n\right) \tag{8}$$

Table 1   All Cases of Positional Relationship between h_{ij} and e_j

Cases		Positional relationship between h_{ij} and e_j
Case 1	$\overline{\gamma}_{ij}^{U} < e_{j}^{U}$	
Case 2	$e_{j}^{L} < \underline{\gamma}_{ij}^{L}$	

Case 3	$\gamma_{ij}^{(k)} \subset \left[e_j^L, e_j^U \right]$ for all k	
Case 4	$\gamma_{ij}^{(k_1)}, \gamma_{ij}^{(k_2)} \not\subset \left[e_j^L, e_j^U \right], k_1 < k_2$, $k_1, k_2 \in \{1, 2, \cdots, K\}$ and $\left[e_j^L, e_j^U \right] \not\subset \gamma_{ij}^{(k)}$ for all k	
	$\left[e_j^L, e_j^U \right] \subset \gamma_{ij}^{(k)}, k \in \{1, 2, \cdots,$	

(4) For case 4, $\gamma_{ij}^{(k_1)}, \gamma_{ij}^{(k_2)} \not\subset \left[e_j^L, e_j^U \right]$ with $k_1 < k_2$, $k_1, k_2 \in \{1, 2, \cdots, K\}$

and case 4 has two scenarios (a) and (b). (a) $\left[e_j^L, e_j^U \right] \subset \gamma_{ij}^{(k)}$ for all k, let

$$\tilde{h}_{ij_1} = \left\{ \gamma_{ij}^{(k_1)}(p_{ij}^{(k_1)}) \middle| k_1 = 1, 2, \cdots, A_1 \right\} \text{ and}$$

$$\tilde{h}_{ij_2} = \left\{ \gamma_{ij}^{(k_2)}(p_{ij}^{(k_2)}) \middle| k_2 = A_2, A_2 + 1, \cdots, K \right\} \text{ be two SPIVHFEs, and } A_2 > A_1.$$

(b) $\left[e_j^L, e_j^U \right] \not\subset \gamma_{ij}^{(k)}$ with $k \in \{1, 2, \cdots, K\}$. There are also two scenarios in (b).

(b_1) If $\left| \gamma_{ij}^{(k)L} - e_j^L \right| > \left| \gamma_{ij}^{(k)U} - e_j^U \right|$, let $\tilde{h}_{ij_1} = \left\{ \gamma_{ij}^{(k_1)}(p_{ij}^{(k_1)}) \middle| k_1 = 1, 2, \cdots, k \right\}$ and

$$\tilde{h}_{ij_2} = \left\{ \gamma_{ij}^{(k_2)}(p_{ij}^{(k_2)}) \middle| k_2 = k+1, k+2, \cdots, K \right\}. (b_2) \text{ If}$$

$\left| \gamma_{ij}^{(k)L} - e_j^L \right| < \left| \gamma_{ij}^{(k)U} - e_j^U \right|$, let $\tilde{h}_{ij_1} = \left\{ \gamma_{ij}^{(k_1)}(p_{ij}^{(k_1)}) \middle| k_1 = 1, 2, \cdots, k-1 \right\}$ and

$$\tilde{h}_{ij_2} = \left\{ \gamma_{ij}^{(k_2)}(p_{ij}^{(k_2)}) \middle| k_2 = k, k+1, \cdots, K \right\}. \text{ Then}$$

$$G_{ij} = d\left(\tilde{h}_{ij_1}, e_j \right), L_{ij} = -d\left(\tilde{h}_{ij_2}, e_j \right), (i = 1, 2, \cdots m; j = 1, 2, \cdots n) \qquad (9)$$

According to the equations (6)-(9), the summary of the formula for gains and losses of the benefit-type attribute is summarized in Table 2. For the cost-type attribute, the specific situation is shown in Table 3.

Table 2 All Cases of Gain and Loss (Benefit-Type Attribute)

Cases		Gain G_{ij}	Loss L_{ij}

Case 1	$\overline{\gamma}_{ij}^{U} < e_j^U$	0	$-d\left(h_{ij}, e_j\right)$
Case 2	$e_j^L < \underline{\gamma}_{ij}^L$	$d\left(h_{ij}, e_j\right)$	0
Case 3	$\gamma_{ij}^{(k)} \subset \left[e_j^L, e_j^U\right]$ for all k	0	0
Case 4	$\gamma_{ij}^{k_1}, \gamma_{ij}^{k_2} \not\subset \left[e_j^L, e_j^U\right], k_1 < k_2,$ $k_1, k_2 \in \{1, 2, \cdots, K\}$	$d\left(\tilde{h}_{ij_2}, e_j\right)$	$-d\left(\tilde{h}_{ij_1}, e_j\right)$

Table 3 All Cases of Gain and Loss (Cost-Type Attribute)

Cases		Gain G_{ij}	Loss L_{ij}
Case 1	$\overline{\gamma}_{ij}^{U} < e_j^U$	$d\left(h_{ij}, e_j\right)$	0
Case 2	$e_j^L < \underline{\gamma}_{ij}^L$	0	$-d\left(h_{ij}, e_j\right)$
Case 3	$\gamma_{ij}^{(k)} \subset \left[e_j^L, e_j^U\right]$ for all k	0	0
Case 4	$\gamma_{ij}^{k_1}, \gamma_{ij}^{k_2} \not\subset \left[e_j^L, e_j^U\right], k_1 < k_2,$ $k_1, k_2 \in \{1, 2, \cdots, K\}$	$d\left(\tilde{h}_{ij_1}, e_j\right)$	$-d\left(\tilde{h}_{ij_2}, e_j\right)$

4. Expansion Method of Gra for Incomplete Weight Information in Pivhfe

In the traditional GRA method, it is necessary to determine the reference point. In this paper, since we need to consider the behavioral factors, we will determine the reference point according to the psychological interval reference point. Next, we will determine the optimal weight according to the expansion method of GRA.

Let $Z = \{Z_1, Z_2, \cdots, Z_n\}$ be a set of alternatives and $Y = \{Y_1, Y_2, \cdots, Y_m\}$ be the set of attributes, $\omega = \{\omega_1, \omega_2, \cdots, \omega_m\}$ is the weight vector of the attribute $Y_j (j = 1, 2, \cdots, m)$, where $\omega_j \in [0,1]$ with $\sum_{j=1}^{m} \omega_j = 1$. In the actual decision making process, it is often encountered that the attribute weight part is known. In general, some known weight information can be expressed as set A [13-16]:

$$A = \left\{\omega_i - \omega_j \geq \varepsilon_i > 0; \omega_i \geq \sigma_i \omega_j, 0 \leq \sigma_i \leq 1; \delta_j \geq \omega_j \geq \delta_j + \varepsilon_j, 0 \leq \delta_j < \right.$$

$$\delta_j + \varepsilon_j \le 1; \omega_i - \omega_j \ge \omega_k - \omega_l; i \ne j \ne k \ne l \big\}$$. Suppose

$$R = \left(h_{ij} \right)_{n \times m} = \left(\gamma_{ij}^{(k)} \left(p_{ij}^{(k)} \right) \right)_{n \times m} \left(k = 1, 2, \cdots, K \right)$$ is the probability

interval-valued hesitant fuzzy decision matrix, where $\gamma_{ij}^{(k)}$ is an interval attribute

evaluation value by decision maker respect to the attribute Y_j .

Next, we will extend the GRA method to solve the weight problem with incomplete weight information.

(1) According to the psychological interval reference point $e_j = \left[e_j^L, e_j^U \right]$ and equations (6)-(9), the reference positive ideal solution (RPIS) and the reference negative ideal solution (RNIS) are determined on the basis of the PIVHFE.

$$h^+ = \left(h_1^+, h_2^+, \cdots, h_m^+ \right), \qquad h^- = \left(h_1^-, h_2^-, \cdots, h_m^- \right) \qquad (10)$$

where $h_j^+ = h_{lj}$ if $v\left(h_{lj}, e_j \right) = \max_i v\left(h_{ij}, e_j \right)$ and $h_j^- = h_{gj}$ if

$v\left(h_{gj}, e_j \right) = \min_i v\left(h_{ij}, e_j \right)$, $v\left(h_{ij}, e_j \right)$ is the prospect value of evaluation value h_{lj} from reference point e_j.

(2) Calculate the grey relational coefficient of each alternative from RPIS and RNIS using the following equation, respectively. The grey relational coefficient of each alternative from RPIS is given as

$$\eta_{ij}^+ = \frac{\min\limits_{1 \le i \le n} \min\limits_{1 \le j \le m} d\left(h_j^+, h_{ij} \right) + \theta \max\limits_{1 \le i \le n} \max\limits_{1 \le j \le m} d\left(h_j^+, h_{ij} \right)}{d\left(h_j^+, h_{ij} \right) + \max\limits_{1 \le i \le n} \max\limits_{1 \le j \le m} d\left(h_j^+, h_{ij} \right)} \qquad (11)$$

The grey relational coefficient of each alternative from RNIS is given as

$$\eta_{ij}^- = \frac{\min\limits_{1 \le i \le n} \min\limits_{1 \le j \le m} d\left(h_j^-, h_{ij} \right) + \theta \max\limits_{1 \le i \le n} \max\limits_{1 \le j \le m} d\left(h_j^-, h_{ij} \right)}{d\left(h_j^-, h_{ij} \right) + \max\limits_{1 \le i \le n} \max\limits_{1 \le j \le m} d\left(h_j^-, h_{ij} \right)} \qquad (12)$$

where the identification coefficient $\theta = 0.5$.

(3) The comprehensive grey correlation coefficients of each alternative are calculated as follows:

$$\eta_i^+ = \sum_{j=1}^{m} \omega_j \eta_{ij}^+, \ i = 1, 2, \cdots, n \tag{13}$$

$$\eta_i^- = \sum_{j=1}^{m} \omega_j \eta_{ij}^-, \ i = 1, 2, \cdots, n \tag{14}$$

(4) According to the principle of GRA method, an alternative is best if it has the maximum comprehensive correlation coefficient from RPIS and the minimum negative correlation coefficient from RNIS. The following equation can be obtained for this model:

$$M_1 : \begin{cases} \min \eta_i^- = \sum_{j=1}^{m} \omega_j \eta_{ij}^-, i = 1, 2, \cdots, n \\ \min \eta_i^+ = \sum_{j=1}^{m} \omega_j \eta_{ij}^+, i = 1, 2, \cdots, n \end{cases} \tag{15}$$

where $\omega_j \in A$.

M_1 can be transformed into the following single-objective optimization model:

$$M_2 : \min \eta(\omega) = \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_j \left(\eta_{ij}^- - \eta_{ij}^+ \right) \tag{16}$$

where $\omega_j \in A$. When solving for model M_2, we can get the optimal weight vector, which is used for the decision attribute.

5. Topsis Method with Interval Reference Point and Incomplete Weight Information

The decision steps of MCDM problem are as follows:

Step 1. Construct value matrix $VM = \left(v_{ij} \right)_{n \times m}$. G_{ij} and L_{ij} are given by the equations (6)-(9) and calculate the prospect value v_{ij} according to equation (1).

Step 2. The optimal weight vector $\omega = \{ \omega_1, \omega_2, \cdots, \omega_m \}$ of each attribute with respect to each alternative is calculated by equations (10)-(16).

Step 3. Calculate the weighted normalized diecision matrix $R = \left(r_{ij} \right)_{n \times m}$ with $r_{ij} = \omega_j v_{ij}$.

Step 4. Determine positive ideal solutions (PIS) and negative ideal solutions (NIS):

$$R^+ = \left\{ r_1^+, r_2^+, \cdots, r_m^+ \right\} = \left\{ \max_j r_{ij} \,\middle|\, j \in B_1, \min_j r_{ij} \,\middle|\, j \in B_2 \right\}$$

$$R^- = \left\{ r_1^-, r_2^-, \cdots, r_m^- \right\} = \left\{ \min_j r_{ij} \,\middle|\, j \in B_1, \max_j r_{ij} \,\middle|\, j \in B_2 \right\}$$

where B_1 and B_2 are the set of benefit-type attribute and cost-type attribute, respectively.

Step 5. The Euclidean distances between each alternative from the positive ideal solution and the negative ideal solution is calculated, respectively:

$$D_i^+ = \sqrt{\sum_{j=1}^m \left(r_{ij} - r_j^+ \right)^2}, \quad D_i^- = \sqrt{\sum_{j=1}^m \left(r_{ij} - r_j^- \right)^2}, \quad i = 1, 2, \cdots, n.$$

Step 6. Calculate the relative closeness RC_i of alternative $Z_i \left(i = 1, 2, \cdots, n \right)$:

$$RC_i = \frac{D_i^-}{D_i^- + D_i^+}$$

Step 7. Rank the alternatives according to the relative closeness RC_i obtained in Step 7, the greater the relative closeness RC_i is, the better alternative Z_i is.

6. A Case Study: Selecting an Optimal Investment Company

6.1 Background

In this section, we apply the methods given above to an example adapted from the example in literature [12].

Suppose an investment company wants to make an investment and requires the best investment choice, there are six investment choices as follows:

(1) Z_1 is a car industry.

(2) Z_2 is a food company.

(3) Z_3 is a computer company.

(4) Z_4 is an arms company.

(5) Z_5 is an TV company.

(6) Z_6 is an refrigerator company.

The company select investments according to the following four attributes:

(1) Y_1 is the risk analysis.

(2) Y_2 is the growth analysis.

(3) Y_3 is the social–political impact analysis.

(4) Y_4 is the environmental impact analysis.

Table 4 The Evaluation Result about Six Alternatives with Respect to Different Attributes

	Y_1	Y_2
Z_1	{[0.6,0.7](1)}	{[0.3,0.4](0.4), [0.4,0.5](0.6)}
Z_2	{[0.3,0.4](0.3), [0.4,0.5](0.7)}	{[0.3,0.4](0.2), [0.5,0.6](0.3), [0.7,0.8](0.5)}
Z_3	{[0.3,0.4](0.5), [0.5,0.6](0.3), [0.7,0.8](0.2)}	{ [0.4,0.5](0.6), [0.6,0.7](0.4)}
Z_4	{[0.4,0.5](0.7), [0.6,0.7](0.3)}	{[0.5,0.6](1)}
Z_5	{[0.7,0.8](1)}	{[0.2,0.3](0.5), [0.4,0.6](0.5)}
Z_6	{[0.3,0.5](1)}	{[0.2,0.3](0.6), [0.3,0.5](0.4)}
	Y_3	Y_4
Z_1	{[0.4,0.5](0.4), [0.6,0.7](0.6)}	{[0.3, 0.5](1)}
Z_2	{[0.5,0.6](0.6), [0.7,0.8](0.4)}	{[0.1,0.2](0.2), [0.3,0.5](0.8)}
Z_3	{[0.2,0.4](1) }	{[0.2,0.3](0.3), [0.4,0.5](0.7)}
Z_4	{[0.1,0.3](1) }	{[0.2,0.4](0.5), [0.5,0.6](0.5)}
Z_5	{[0.2,0.3](0.6), [0.4,0.6](0.4)}	{[0.7,0.8](1)}
Z_6	{[0.5,0.6](1)}	{[0.2,0.3](1)}

Table 5 The Psychological Interval Reference Point of Attributes

	Y_1	Y_2	Y_3	Y_4
The psychological interval reference point	[0.4,0.6]	[0.4,0.5]	[0.4,0.6]	[0.3,0.5]

In the attribute $Y = \{Y_1, Y_2, Y_3, Y_4\}$, Y_1 is the cost-type attribute and the others are the benefit-type attribute. The information about the attribute weights is

partly known: $A =$
$\{0.15 \leq \omega_1 \leq 0.3, 0.1 \leq \omega_2 \leq 0.2, 0.3 \leq \omega_3 \leq 0.4, 0.08 \leq \omega_4 \leq 0.11\}$ $\omega_j \geq 0$,

$\sum_{j=1}^{4} \omega_j = 1$. PIVHFE can be used to evaluate every attribute of all investment projects, and the evaluation result of decision maker is shown in Table 4. Meanwhile, the decision maker's psychological interval reference point for each attribute is shown in Table 5. In the following calculation, we take $a = 0.5$ and $q = 1$.

6.2 Topsis Method with Interval Reference Point and Incomplete Weight Information

Step 1. According to Table 4, Table 5 and equations (1), (6)-(9), the value matrix $VM = \left(v_{ij} \right)_{n \times m}$ can be obtained.

$$VM = \begin{bmatrix} -0.424 & -0.132 & 0.120 & 0 \\ 0.065 & 0.133 & 0.132 & -0.161 \\ -0.059 & 0.108 & -0.546 & -0.147 \\ -0.147 & 0.132 & -0.780 & -0.059 \\ -0.664 & -0.258 & -0.424 & 0.397 \\ 0.132 & -0.374 & 0 & -0.424 \end{bmatrix}$$

Step 2. According to the incomplete information given by set A, the optimal weight vector is calculated according to formulas (10)-(16). First, the evaluation values in Table 4 are adjusted according to method of Example 2. The results of adjustment are shown in Table 6.

The following objective programming model is established by using model M_2:

$$\min \eta\left(\omega \right) = -0.5201\omega_1 + 0.2473\omega_2 - 0.1912\omega_3 + 1.16\omega_4$$

where $\omega \in A$. Solve this model to get the optimal weight vector is
$$\omega = \{0.3, 0.2, 0.4, 0.1\}$$.

Step 3. The weight normalization decision matrix R is obtained from the optimal weight vector.

$$R = \begin{bmatrix} -0.1272 & -0.0264 & 0.0480 & 0 \\ 0.0195 & 0.0266 & 0.0528 & -0.0161 \\ -0.0177 & 0.0216 & -0.2184 & -0.0147 \\ -0.0441 & 0.0264 & -0.3120 & -0.0059 \\ -0.1992 & -0.0516 & -0.1696 & 0.0397 \\ 0.0396 & -0.0748 & 0 & -0.0424 \end{bmatrix}$$

Step 4. According to the weight normalization decision matrix obtained in the Step 4, the PIS and NIS are determined as

$$R^{+} = \left(-0.1992, 0.0266, 0.0528, 0.0397 \right) \text{ and } R^{-} = \left(0.0396, -0.0748, -0.3120, -0.0424 \right), \text{ respectively.}$$

Step 5. The Euclidean distance of each alternative from the PIS and NIS is calculated, respectively.

Table 6 : the Adjusted Evaluation Result

	Y_1		Y_2	
Z_1	{[0.6,0.7](0.3), [0.6,0.7](0.2), [0.6,0.7](0.1), [0.6,0.7](0.2)}	[0.6,0.7](0.2),	{[0.3,0.4](0.2), [0.4,0.5](0.1), [0.4,0.5](0.4)}	[0.3,0.4](0.2), [0.4,0.5](0.1),
Z_2	{[0.3,0.4](0.3), [0.4,0.5](0.2), [0.4,0.5](0.2)}	[0.4,0.5](0.2), [0.4,0.5](0.1),	{[0.3,0.4](0.2), [0.5,0.6](0.1), [0.7,0.8](0.4)}	[0.5,0.6](0.2), [0.7,0.8](0.1),
Z_3	{[0.3,0.4](0.3), [0.5,0.6](0.2), [0.7,0.8](0.2)}	[0.3,0.4](0.2), [0.5,0.6](0.1)	{[0.4,0.5](0.2), [0.4,0.5](0.1), [0.6,0.7](0.4)}	[0.4,0.5](0.2), [0.4,0.5](0.1),
Z_4	{[0.4,0.5](0.3), [0.4,0.5](0.2), [0.6,0.7](0.2)}	[0.4,0.5](0.2), [0.6,0.7](0.1),	{[0.5,0.6](0.2), [0.5,0.6](0.1), [0.5,0.6](0.4)}	[0.5,0.6](0.2), [0.5,0.6](0.1),
Z_5	{[0.7,0.8](0.3), [0.7,0.8](0.2), [0.7,0.8](0.2)}	[0.7,0.8](0.2), [0.7,0.8](0.1),	{[0.2,0.3](0.2), [0.2,0.3](0.1), [0.4,0.6](0.4)}	[0.2,0.3](0.2), [0.4,0.6](0.1),
Z_6	{[0.3,0.5](0.3), [0.3,0.5](0.2), [0.3,0.5](0.2)}	[0.3,0.5](0.2), [0.3,0.5](0.1),	{[0.2,0.3](0.2), [0.2,0.3](0.1), [0.3,0.5](0.4)}	[0.2,0.3](0.2), [0.2,0.3](0.1),
	Y_3		Y_4	
Z_1	{[0.4,0.5](0.4), [0.6,0.7](0.6)}	[0.6,0.7](0.2),	{[0.3,0.5](0.2), [0.3,0.5](0.2), [0.3,0.5](0.5)}	[0.3,0.5](0.1),
Z_2	{[0.5,0.6](0.4), [0.7,0.8](0.4)}	[0.5,0.6](0.2),	{[0.1,0.2](0.2), [0.3,0.5](0.2), [0.3,0.5](0.5)}	[0.3,0.5](0.1),
Z_3	{[0.2,0.4](0.4), [0.2,0.4](0.4)}	[0.2,0.4](0.2),	{[0.2,0.3](0.2), [0.4,0.5](0.2), [0.4,0.5](0.5)}	[0.2,0.3](0.1),
Z_4	{[0.1,0.3](0.4),	[0.1,0.3](0.2),	{[0.2,0.4](0.2),	[0.2,0.4](0.1),

	[0.1,0.3](0.4)}	[0.2,0.4](0.2), [0.5,0.6](0.5)}
Z_5	{[0.2,0.3](0.4), [0.2,0.3](0.2), [0.4,0.6](0.4)}	{[0.7,0.8](0.2), [0.7,0.8](0.1), [0.7,0.8](0.2), [0.7,0.8](0.5)}
Z_6	{[0.5,0.6](0.4), [0.5,0.6](0.2), [0.5,0.6](0.4)}	{[0.2,0.3](0.2), [0.2,0.3](0.1), [0.2,0.3](0.2), [0.2,0.3](0.5)}

Table 7 : the Distance of Each Alternative from the Pis and Nis

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
D_i^+	0.0979	0.2257	0.3309	0.3990	0.2357	0.2772
D_i^-	0.4019	0.3801	0.1487	0.1363	0.2908	0.3120

Step 6. Calculate the relative closeness RC_i of alternative $Z_i (i = 1, 2, \cdots, n)$.

Table 8 : the Relative Closeness of Each Alternative

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
RC_i	0.8041	0.6274	0.3100	0.2546	0.5523	0.5295

Step 7. Table 8 shows that $RC_1 > RC_2 > RC_5 > RC_6 > RC_3 > RC_4$, so alternative Z_1 is the best investment alternative.

6.3 Sensitivity Analysis and Stability Analysis

In this subsection, we performed sensitivity analysis and stability analysis for parameter q in distance measure and parameter λ in the proposed method.
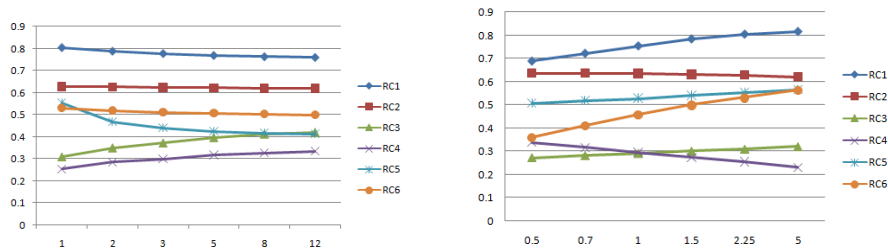

Figure.4 The Calculation Results with Different q.   Figure 4: the Calculation Results with Different λ.

First, we analyze the parameter q of distance measure. Let $q = 1, 2, 3, 5, 8, 12$ and $\alpha = \beta = 0.88$, $\lambda = 2.25$, and analyze the influence of the change of parameter q. The proposed method is used to calculate the relative closeness RC_i of each alternative and rank them. Figure 3 shows the influence of different parameters q on the results. As can be seen from Figure 3, when the parameter q changes from 1 to 2, although the ranking has changed a little bit, the ranking of Z_1 does not change, so it does not affect the choice of the best alternative. When $q \geq 5$, the relative closeness hardly changes at all, and the best alternative is always Z_1. This result indicates that the final ranking of alternatives is stable with the change of parameter q.

Similar, we discuss the influence of parameter λ in S-shapes value function. When $q = 1$ and $\alpha = \beta = 0.88$, let $\lambda = 0.5, 0.7, 1, 1.5, 2.25, 5$, the ranking results are shown in Figure 4. When $\lambda = 0.5$, relative closeness is ranked, and the result is $RC_1 > RC_2 > RC_5 > RC_6 > RC_4 > RC_3$. When $\lambda = 1$, RC_3 and RC_4 are almost equal, while when $\lambda > 1$, $RC_3 > RC_4$. With the increase of λ, the gap between RC_3 and RC_4 is increasing, and $RC_3 > RC_4$ are always true. Although there are some changes in the ranking, RC_1 is always the largest and the difference with other relative closeness is increasing with the change of λ, that is, Z_1 is always the best alternative. The sensitivity and stability of the proposed method are discussed above.

7. Conclusions

In this paper, we introduce probabilistic interval-valued hesitant fuzzy sets and probabilistic interval-valued hesitant fuzzy elements, as well as define new related properties and operations. Then, we define the hybrid distance between PIVHFE and the interval fuzzy number and the distance between two PIVHFEs. At the same time, we expand the GRA method to solve the problem of incomplete weight information. Since the traditional TOPSIS method does not take into account the decision maker's psychological behavior, we developed the TOPSIS method with psychological reference point. Finally, the proposed method is applied to an example, and the sensitivity analysis and stability analysis of the proposed method are also discussed. Future study will focus on consistency based on prospect theory.

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