# Optimal decision rules acquisition for interval-valued ordered decision information systems

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***ABSTRACT.*** *The interval is viewed as basic knowledge granule and used to define the lower and upper approximations. This model is applied to interval-valued ordered decision information systems, and used to induce useful "at least and at most" decision rules. To obtain optimal decision rules, the concept of relative reducts of an interval is proposed, and the corresponding discernibility function is constructed for computing the relative ruduct.*

***Keywords****: rough set; interval knowledge granules; relative reduct; decision rule*

## 1. Introduction

Because of limited discriminatory power of the criteria and hesitation of the decision maker, dominance principle is often violated for some objects [1]. In this case, Pawlak rough set model [2], which concerns the discernibility between objects, can not cope with the inconsistency in an ODIS. So Greco et al[3] proposed the dominance-based rough set approach (DRSA) to solve the inconsistency problem. They also gave an algorithm for computing minimal decision rules[1]. For the attribute reduction problem in the ODIS, Susmaga et al[4] proposed reducts preserving an information measure called a quality of sorting. Inuiguchi, Yoshioka and Kusunoki [5] proposed several kinds of reducts and clarified relations among the proposed reducts and previous ones. Moreover, the DRSA has been extended to variable consistency dominance-based rough set approach[6], variable precision dominance-based rough set approach[7], and stochastic dominance-based rough set approach [8]. The DRSA was also applied to decision rules acquisition and attribute reduction for varieties of other ODISs, such as incomplete ODISs [9], interval ODISs [10], and set-valued ODISs [11].

Both the DRSA and its extended models take the dominating classes and dominated classes as basic knowledge granules. These models were successfully used to extract “at least” or “at most” decision rules. To obtain “at least and at most” decision rule, Guan et al[12] take “interval”, an intersection of the dominating class of one object and the dominated class of another object, as a basic knowledge granule, and use intervals to define the lower and upper approximations.

In this paper, we will apply the “interval” to the interval-valued ordered decision information system (IVODIS). In Section 2, some notations and basic concepts for the ODIS and the original DRSA are introduced. Also, we briefly introduce the I-DRSA. In section 3, we apply the I-DRSA to the IVODIS. We discuss relative reducts and optimize “at least and at most” decision rules acqusition for based on interval knowledge granules. Finally, we conclude our work in Section 4.

## 2. The original dominance-based rough set approach

### *2.1 The ordered decision information system (ODIS)*

An information system is a quadruple $S=(U,AT,V,f)$, where $U$ is called the universe of discourse; $AT$ is a finite set of attributes; $V=\cup_{a\in AT}V_a$ with $V_a$ being the domain of attribute $a$; $f$ is an information function satisfying $f(x,a)\in V_a$. Denote $f(x,a)=a(x)$ for simplicity.

In general, if the attributes contained in an information system are classified into the condition attributes($C=\{c_1,c_2,\cdots,c_n\}$)and decision attributes($D$), then it is called a decision information system (DIS). In this case, $AT=C\cup D$ and $C\cap D=\varnothing$. Without loss of generality, assume that $D=\{d\}$. The partition of $U$

determined by $d$ is denoted as $\{Cl_1, Cl_2, \cdots, Cl_r\}$.

**Definition 1[3]** In a decision information system (DIS), if all the condition attributes are criteria, then it is called an ordered decision information system (ODIS) or an ordered decision table (ODT).

### *2.2 The original dominance-based rough set approach (DRSA)*

In an ODIS, the domain of a criterion $a \in C$ is completely preordered by an outranking relation $\geq_a$. For $x, y \in U$, if $x$ is at least as good as $y$ with respect to criterion $a$, then it is denoted as $x \geq_a y$ or $y \leq_a x$. For $B \subseteq C$, denote dominance relation as follows:

$$R_B^{\geq} = \{(x, y) \in U \times U \mid x \geq_b y, \forall b \in B\},$$

$$R_B^{\leq} = \{(x, y) \in U \times U \mid x \leq_b y, \forall b \in B\}$$

Denote $[x]_B^{\geq} = \{y \in U \mid (y, x) \in R_B^{\geq}\}$, $[x]_B^{\leq} = \{y \in U \mid (y, x) \in R_B^{\leq}\}$,

then $[x]_B^{\geq}$ and $[x]_B^{\leq}$ are called dominating class and dominated class of $x$.

**Definition 2[3]** In an ODIS $S = (U, C \cup \{d\}, V, f)$, for $X \subseteq U$ and $B \subseteq C$, let

$$\underline{R_B^{\geq}}(X) = \{x \mid x \in U, [x]_B^{\geq} \subseteq X\}, \quad \overline{R_B^{\geq}}(X) = \{x \mid x \in U, [x]_B^{\leq} \cap X \neq \emptyset\},$$

$\underline{R_B^{\geq}}(X)$ and $\overline{R_B^{\geq}}(X)$ are called the lower and the upperapproximation approximation of $X$.

### *2.3 Interval Knowledge Granules for DRSA(I-DRSA)*

For extracting the “at least and at most” decision rules with the decision part $s \le d \le t$, where $s, t \in V_d$, we proposed a new type of knowledge granules, called “interval”[12].

**Definition 3[12]** In an ODIS $S = (U, C \cup \{d\}, V, f)$, for $B \subseteq C$ and $(x_j, x_i) \in R_B^{\geq}$, denote $[x_i, x_j]_B = [x_i]_B^{\geq} \cap [x_j]_B^{\leq} = \{y \mid y \in U, x_i \le_B y \le_B x_j\}$ .We call $[x_i, x_j]_B$ an interval determined by $(x_j, x_i) \in R_B^{\geq}$ . For $(x_j, x_i) \notin R_B^{\geq}$ , let $[x_i, x_j]_B = \varnothing$.

Taking intervals as basic knowledge granules, we can define the lower and upper approximations of $X \subseteq U$ as follows.

**Definition 4[12]** In an ODIS $S = (U, C \cup \{d\}, V, f)$, for $X \subseteq U$ and $B \subseteq C$, let

$$\underline{R_B^I}(X) = \{(x_j, x_i) \mid (x_j, x_i) \in R_B^{\geq}, [x_i, x_j]_B \subseteq X\},$$

$$\overline{R_B^I}(X) = \{(x_j, x_i) \mid (x_j, x_i) \in R_B^{\geq}, [x_i, x_j]_B \cap X \neq \varnothing\},$$

We cal $\underline{R_B^I}(X)$ and $\overline{R_B^I}(X)$ the Interval-lower and Interval-upper approximation of $X$ .

Using the I-lower approximations of $Cl_s^t$, from $(x_j, x_i) \in \underline{R_B^{\leq\geq}}(Cl_s^t)$, we can inducing the certain “at least and at most” decision rules as follows:

if $x \in [x_i, x_j]_B$, then $x \in Cl_s^t$ .Or if $\wedge_{b \in B}(b(x_i) \le, b, \le b(x_j)) \to (s \le, d, \le t)$.

## 3. Optimal decision rules acquisition in the IVODIS based on interval knowledge granules

### *3.1 I-DRSA established in the IVODIS*

An ordered decision information system is called interval-valued ordered decision information system(IVODIS), if $V_a$ is a set of interval-valued numbers. We denote the interval number of $x$ under the attribute $a$ as follows: $f(x,a)=a(x)=[a(x^m),a(x^M)]$. The following Table 1 presents an IVODIS.

*Table 1 An interval-valued ordered decision information system*

| $U$ | $c_1$ | $c_2$ | $c_3$ | $C_4$ | $C_5$ | $d$ |
|---|---|---|---|---|---|---|
| $x_1$ | [2, 2.4] | [1.5, 3] | [4, 4.8] | [3, 3.6] | [6,7.2] | 3 |
| $x_2$ | [2.8,4] | [2.1,2.7] | [4.8,7.2] | [3.6,4.8] | [7.2,10.8] | 2 |
| $x_3$ | 2.4 | [1.2,1.8] | [4.8,7.2] | [2.4,3.6] | [4.8,6.0] | 3 |
| $x_4$ | [1.2,1.6] | [0.9,1.2] | 2.4 | [1.8, 2.4] | [3.6,4.8] | 2 |
| $x_5$ | [0.8,4] | [2.1,3] | [4.8,7.2] | [1.2,4.8] | [2.4 3.6] | 1 |
| $x_6$ | [2, 2.8] | [1.5,3] | [4.8,6.4] | [1.2,6] | [7.2,8.4] | 3 |
| $x_7$ | [2, 2.4] | 1.8 | [3.2,8] | [2.4,5.4] | [6,7.2] | 1 |
| $x_8$ | [2.8,3.2] | [1.5, 2.4] | [4.8,8] | [3.6,6] | [7.2,10.8] | 4 |

**Definition 5** In an IVODIS, for $B\subseteq C$ and $(x_j,x_i)\in R_B^{\geq}$, denote

$$[x_i,x_j]_B=[x_i]_B^{\geq}\cap[x_j]_B^{\leq}=\{y\mid y\in U, x_i^m\leq_B y^m\leq_B x_j^m \text{ and } x_i^M\leq_B y^M\leq_B x_j^M\}.$$

We call $[x_i,x_j]_B$ an interval determined by $(x_j,x_i)\in R_B^{\geq}$.

We will take intervals as basic knowledge granules to define the lower and upper approximations of $X\subseteq U$ in the following .

**Definition 6** In an IVODIS $S=(U,C\cup\{d\},V,f)$, for $X\subseteq U$ and $B\subseteq C$, let

$$\underline{R_B^{\leq\geq}}(X)=\{(x_j,x_i)\mid(x_j,x_i)\in R_B^{\geq},[x_i,x_j]_B\subseteq X\},$$

$\overline{R_B^{\leq\geq}}(X)=\{(x_j,x_i)\,|\,(x_j,x_i)\in R_B^{\geq},[x_i,x_j]_B\cap X\neq\varnothing\}$.

We call $\underline{R_B^{\leq\geq}}(X)$ and $\overline{R_B^{\leq\geq}}(X)$ the I-lower and I-upper approximation of $X$ with respect to $B$.

Let

$m\underline{R_B^{\leq\geq}}(Cl_s^t)=\{(x_j,x_i)\,|\,(x_j,x_i)\in\underline{R_B^{\leq\geq}}(Cl_s^t),\ [x_i,x_j]_B\not\subset[x_p,x_q]_B \text{ for } \forall\,(x_q,x_p)\in\underline{R_B^{\leq\geq}}(Cl_s^t)\}$.

Then $\forall(x_j,x_i)\in m\underline{R_B^{\leq\geq}}(Cl_s^t)$ generates a minimal interval decision rule with $s\leq d\leq t$.

**Example 1** In the IVODIS presented in the Table 1, we have

$[x_1]_C^{\geq}=\{x_1\}$, $[x_2]_C^{\geq}=\{x_2\}$, $[x_3]_C^{\geq}=\{x_2,x_3,x_8\}$, $[x_4]_C^{\geq}=\{x_1,x_2,x_3,x_4,x_7,x_8\}$, $[x_5]_C^{\geq}=\{x_5\}$,

$[x_6]_C^{\geq}=\{x_6\}$, $[x_7]_C^{\geq}=\{x_7\}$, $[x_8]_C^{\geq}=\{x_8\}$, $[x_1]_C^{\leq}=\{x_1,x_4\}$, $[x_2]_C^{\leq}=\{x_2,x_3,x_4\}$, $[x_3]_C^{\leq}=\{x_3,x_4\}$,

$[x_4]_C^{\leq}=\{x_4\}$, $[x_5]_C^{\leq}=\{x_5\}$, $[x_6]_C^{\leq}=\{x_6\}$, $[x_7]_C^{\leq}=\{x_4,x_7\}$, $[x_8]_C^{\leq}=\{x_3,x_4,x_8\}$.

$\underline{R_C^{\leq\geq}}(Cl_2^3)=\{(x_3,x_2),(x_4,x_1),(x_4,x_2),(x_4,x_3),(x_1,x_1),(x_2,x_2),(x_3,x_3),(x_4,x_4),(x_6,x_6)\}$,

$m\underline{R_C^{*\leq\geq}}(Cl_2^3)=\{(x_4,x_1),(x_4,x_2)\}$.

$(x_4,x_2)$ generates a minimal $D_C^{\leq\geq}$-decision rule with decision part $2\leq d\leq 3$:

$([1.2,1,6]\leq, c_1, \leq[2.8,4])\wedge([0.9,1.2]\leq, c_2, \leq[2.1,2.7])\wedge(2.4\leq, c_3, \leq[4.8,7.2])\wedge([1.8,2.4]\leq, c_4, \leq[3.6,4.8])$

$\wedge([3.6,4.8]\leq, c_5, \leq[7.2,10.8])\rightarrow(2\leq, d, \leq 3)$,

***3.2 Retive reducts and optimal decison rules acqusition in IVODIS***

To compute the optimal decision rules, we will firstly discuss the reducts of the interval $[x_i, x_j]_C$ defined below.

**Definition 7** For $(x_j, x_i)\in \underline{R_C^{\geq}}(Cl_s^t)$ and $B\subseteq C$, if $B$ is the miniman subset satisfy $d[x_i, x_j]_B = d[x_i, x_j]_C$, then $B$ is called a relative reduct of $[x_i, x_j]_C$ with respect to $Cl_s^t$.

$B$ is a relative ruduct of $[x_i, x_j]_C$ if and only if $\wedge_{c\in B}(c(x_i)\leq, c\ ,\leq c(x_j))\rightarrow(s\leq, d, \leq t)$ is an optimal decision rule of $\wedge_{c\in C}(c(x_i)\leq, c\ ,\leq c(x_j))\rightarrow(s\leq, d, \leq t)$.

Therefore, by the reducts of $[x_i, x_j]_C$, we can obtain all the optimal decision rules supported by objects in $[x_i, x_j]_C$. To compute reducts of $[x_i, x_j]_C$, we firstly give the judgment theorem.

**Theorem 1** For $(x_j, x_i)\in R_C^{\geq}$ and $B\subseteq C$, $d[x_i, x_j]_B = d[x_i, x_j]_C \Leftrightarrow$ $\alpha(y, x_i)\cap B\neq\varnothing$ or $\alpha(x_j, y)\cap B\neq\varnothing$ for any $y$ such that $d(y)\notin d[x_i, x_j]_C$, where $\alpha(x, y)=\{b\in C \mid b(x)\ngeq b(y)\}$

$= \{b \in C \mid b(x^M) < b(y^M)\, or\, b(x^m) < b(y^m)\}$ and $d(X) = \{f(x,d) \mid x \in X\}$ .

**Proof.** "$\Rightarrow$" : Assume there exists $y \in U$ such that $dy \notin d[x_i, x_j]_C$ , both $\alpha(y,x_i) \cap B = \varnothing$ and $\alpha(x_j, y) \cap B = \varnothing$ are satisfied. Then, $\forall b \in B$ , $b(y) \geq b(x_i)$ as well as $b(x_j) \geq b(y)$ . So we have $y \in [x_i, x_j]_B$ . By the conditional assumption we can obtain $d(y) \in d[x_i, x_j]_B = d[x_i, x_j]_C$ , which is contradictive to $d(y) \notin d[x_i, x_j]_C$ . This indicates that $\alpha(y,x_i) \cap B \neq \varnothing$ or $\alpha(x_j, y) \cap B \neq \varnothing$ for any $y$ such that $d(y) \notin d[x_i, x_j]_C$ .

"$\Leftarrow$": Assume that $d[x_i, x_j]_B \neq d[x_i, x_j]_C$ , then there exists at least one object $k \in d[x_i, x_j]_B$ such that $k \notin d[x_i, x_j]_C$ . From the conditional assumption we can derive that $\alpha(y,x_i) \cap B \neq \varnothing$ or $\alpha(x_j, y) \cap B \neq \varnothing$ . Hence, there exists at least one attribute $b \in B$ such that $b \in \alpha(y,x_i)$ or $b \in \alpha(x_j, y)$ . So, $y \notin [x_i]_B^{\geq}$ or $y \notin [x_j]_B^{\leq}$ . This is contradictive to $y \in [x_i, x_j]_B$ . Therefore, $d[x_i, x_j]_B = d[x_i, x_j]_C$ must hold.

Based on Theorem 1, we can construct a dominance discernibility function for $[x_i, x_j]_C$ , which helps us compute the relative reducts of the interval $[x_i, x_j]_C$ .

**Definition 8** For $(x_j, x_i) \in \underline{R_C^{\leq}}(Cl_s^t)$ , let

$$\Delta_{[s,t]}^{\leq}([x_i, x_j]_C) = \bigwedge_{d(y) \notin d[x_i, x_j]_C} \{[\vee \alpha(y,x_i)] \vee [\vee \alpha(x_j, y)]\} ,$$

We call $\Delta_{[s,t]}^{\le\ge}([x_i, x_j]_C)$ a dominance discernibility function of $[x_i, x_j]_C$ with respect to $Cl_s^t$.

Based on the Definition 7, 8 and Theorem 1, we obtain the following Proposition 1 by the Boolean reasoning technique [13].

**Proposition 1** For $(x_j, x_i) \in \underline{R_C^{\le\ge}}(Cl_s^t)$ and $B \subseteq C$, we have

$B$ is a relative reduct of interval $[x_i, x_j]_C$ with respect to $Cl_s^t$ if and only if $\wedge B$ is a prime implicant of $\Delta_{[s,t]}^{\le\ge}([x_i, x_j]_C)$, where $\wedge B = \wedge_{b\in B} b$.

**Example 2** (Continued from Example 1) For the IVODIS presented in the Table 1, we have $(x_4, x_2) \in \underline{R_C^{\le\ge}}(Cl_2^3)$, and

$$\Delta_{[2,3]}^{\le\ge}([x_4, x_2]_C) = \wedge_{dy \notin d([x_4, x_2]_C)} \{[\vee\alpha(y, x_4)] \vee [\vee\alpha(x_2, y)]\}$$

$$= (c_1 \vee c_2 \vee c_4 \vee c_5) \wedge (c_3 \vee c_4) = (c_1 \wedge c_3) \vee (c_2 \wedge c_3) \vee (c_3 \wedge c_5) \vee c_4 .$$

So, the reducts of $[x_4, x_2]_C$ with respect to $Cl_2^3$ are $\{c_1, c_3\}$, $\{c_2, c_3\}$, $\{c_3, c_5\}$ and $\{c_4\}$, we obtain the optimal decision rules as follows:

$([1.2,1.6] \le, c_1, \le [2.8,4]) \wedge (2.4 \le, c_3, \le [4.8,7.2]) \to (2 \le, d, \le 3)$,

$([0.9,1.2] \le, c_2, \le [2.1,2.7]) \wedge (2.4 \le, c_3, \le [4.8,7.2]) \to (2 \le, d, \le 3)$,

$(2.4 \le, c_3, \le [4.8,7.2]) \wedge ([3.6,4.8] \le, c_5, \le [7.2,10.8]) \to (2 \le, d, \le 3)$,

$([1.8,2.4] \le, c_4, \le [3.6,4.8]) \to (2 \le, d, \le 3)$.

## 4. Conclusions

We apply interval knowledge granules to interval-valued ordered decision information systems for knowledge reduction and optimal decision rules acquisition. By I-lower approximation, we can induce “at least and at most” decision rules. For rules optimization and attribute reduction, the relative reducts of an interval is proposed, and each reduct can induce an optimal decision rule supported by objects in the interval.

## 5. Ackonwledgements

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